On Some Alternative Formulations of the Euler and Navier–Stokes Equations

by

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Contents

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Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. The work presented was carried out by the author alone, under the supervision of James Robinson.

Parts of this thesis form the basis for the following papers: Pooley (2016), Pooley and Robinson (2016a), and Pooley and Robinson (2016b).

Abstract

In this thesis we study well-posedness problems for certain reformulations and models of the Euler equations and the Navier–Stokes equations. We also prove several global well-posedness results for the diffusive Burgers equations.

We discuss the Eulerian-Lagrangian formulation of the incompressible Euler equations considered by Constantin (2000). Using this formulation we give a new proof that the Euler equations are locally well-posed in $H^s(\mathbb{T}^d)$ for $s > \frac{d}{2} + 1$. Constantin proved a local well-posedness result for this system in the Hölder spaces $C^{1,\mu}$ for $\mu > 0$, but an analysis in Sobolev spaces is perhaps more natural.

After suggesting a possible Eulerian-Lagrangian formulation for the incompressible Navier–Stokes equations in which the back-to-labels map is not diffused, we obtain the formulation written in terms of the so-called magnetization variables, as studied by Montgomery-Smith and Pokorný (2001). We give a rigorous analysis of the equivalence between this formulation and the classical one, in the context of weak solutions. Noting certain similarities between this formulation and the diffusive Burgers equations we begin a study of the latter.

We prove that the diffusive Burgers equations are globally well-posed in $L^p \cap L^2(\Omega)$ for certain domains $\Omega \subseteq \mathbb{R}^d$, $p > d$, and $d = 2$ or $d = 3$. Moreover, we prove a global well-posedness result in $H^{1/2}(\mathbb{T}^3)$.

Lastly, we consider a new model of the Navier–Stokes equations, obtained by modifying one of the nonlinear terms in the magnetization variables formulation. This new system admits a maximum principle and we prove a global well-posedness result in $H^{1/2}(\mathbb{T}^3)$ following our analysis of the Burgers equations.

Chapter 1

Preliminaries

1.1 Introduction

In this thesis, we will discuss certain reformulations of the Euler and Navier–Stokes equations. From one of these reformulations we derive a new model of the Navier–Stokes system, for which we prove global well-posedness in $H^{1/2}(\mathbb{T}^3)$. That model has features of both the Navier–Stokes equations and the diffusive Burgers equations. For this reason, we dedicate two chapters to various (global) existence and uniqueness results for the Burgers equations, with proofs that follow, as closely as possible, familiar treatments of the Navier–Stokes equations.

The bulk of the mathematical content of this thesis is based closely on three papers; two by Pooley and Robinson (2016b,a) (Chapters 3 and 5, respectively) and one by Pooley (2016) (Chapter 6). In Chapter 4, we add several new well-posedness results for the Burgers equations to those in Chapter 5 (and the aforementioned article). These two chapters are ordered by technical level, rather than chronology.

In Chapter 2 we will give introductory remarks on the Euler and Navier–Stokes equations. This includes a (somewhat heuristic) derivation from first principles, and a short discussion of the history and some important known results.

In Chapter 3 we will discuss an interesting reformulation of the Euler equations, which Constantin (2000) has studied. Following Constantin we will refer to this system as the *Eulerian-Lagrangian formulation* since it uses a mixture of Eulerian and Lagrangian coordinates. This system is based on the Weber formula, named after Weber (1868), which in modern notation can be written as

$$u(x,t) = \mathbb{P}((\nabla A)^\top u_0(A(x,t))),$$

where u is a putative solution of the Euler equations and $\mathbb{P}$ denotes the Leray projection (see Section 1.2). Here A denotes a time-dependent transformation $A : \Omega \times [0,T] \to \Omega$, of the domain Ω. It is the so-called *back-to-labels map*, which is essentially the inverse of the trajectories map. Precisely, the Lagrangian trajectory map X, is the solution of

$$\frac{\partial}{\partial t} X(a,t) = u(X(a,t),t),\ X(a,0) = a,$$

and the back-to-labels map is defined to be the solution of

$$A(X(a,t),t) = a,$$

for all $a \in \Omega$ and all t.

Constantin (2000) used the Eulerian-Lagrangian formulation to give a concise local well-posedness proof for the Euler equations in certain Hölder spaces $C^{1,\mu}$. Broadly following his approach, but independent of his result, we prove local well-posedness for the Euler equations with d-dimensional periodic boundary conditions in the Sobolev spaces H^s, where $s > \frac{d}{2} + 1$. Although local existence of solutions follows from Constantin's result (by virtue of Morrey's inequality) it is useful to check that solutions in H^s stay there for a short time. Moreover it is natural to take an approach based on Sobolev norms (or even more general ones) if we hope to find local well-posedness results in larger function spaces (for $d \geq 3$).

Our main iterative construction in Chapter 3 differs from Constantin's; however, we will also give an alternative scheme (valid if, additionally, $s \in \mathbb{Z}$), that more closely resembles his approach. This alternative relies on a lemma estimating certain compositions of Sobolev functions which may be amenable to refinements of independent interest.

At the end of the chapter we briefly derive the magentization variables formulation of the Navier–Stokes equations, which is one of the subjects of Chapter 6. The similarity to the Burgers equations, namely the lack of a pressure term and absense of an incompressibility constraint, motivates our discussions in Chapters 4 and 5.

The Burgers equations are sometimes used as a model for the Navier–Stokes equations and it is interesting to compare corresponding analyses of the two; however, we have not found much discussion of the former system in contexts familiar for the latter.

In Chapters 4 and 5, we will prove that the Burgers equations are globally well-posed for initial data in H^1 for bounded domains $\Omega \subset \mathbb{R}^d$ and the whole space $\mathbb{R}^d$ for $d = 2$ or 3, before extending these results to less regular data. Our analysis in the H^1 case appears to be similar to the work of Heywood and Xie (1997), but we believe our study in the more difficult cases is new.

In detail, for dimensions $d = 2$ and 3, we will prove global well-posedness results in $L^p \cap L^2$ $p > d$ for d-dimensional bounded domains and the whole space $\mathbb{R}^d$. We will also prove a global well-posedness result in the pseudo-critical[1] space $H^{1/2}(\mathbb{T}^3)$. It is worth noting that although a maximum principle makes it relatively straightforward to extend local solutions globally, the short time existence arguments for the Burgers equations are more complicated than those for the Navier–Stokes system. In particular we have neither momentum conservation nor an existence theory for L^2 data, to aid our constructions.

In Chapter 6 we will give further analysis of the magnetization variables formulation of the Navier–Stokes equations, namely

$$\partial_t w + (\mathbb{P}w \cdot \nabla)u + (\nabla \mathbb{P}w)^\top w - \Delta w = 0,$$

which is related to the classical formulation via a Leray projection $u = \mathbb{P}w$.

We will prove that a weak solution $w \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ corresponds to a weak solution of the classical formulation. In this class of functions we can only give a partial converse; however we will show that if $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2}(\mathbb{T}^3))$ is a weak solution of the Navier–Stokes equations, then a unique weak solution w to the above system exists and $u = \mathbb{P}w$.

[1] Of course in $H^{1/2}(\mathbb{T}^3)$, neither the norm nor the domain are invariant under the natural scaling of the Burgers equations. Nonetheless, delicate estimates are required in this case.

We then go on to consider the system

$$\partial_t w + (\mathbb{P}w \cdot \nabla)u + (\nabla w)^\top w - \Delta w = 0,$$

as a new model for the Navier–Stokes equations. Like the Burgers equations, this system admits a maximum principle, moreover momentum is conserved in the corresponding evolution. Accordingly we give a proof that this model is globally well-posed in $H^{1/2}(\mathbb{T}^3)$, following our treatment of the Burgers equations.

The rest of this chapter is devoted to the standard notation and technical tools we will use in this thesis.

1.2 Notation and tools

1.2.1 Constants

In many places where we make an estimate we will chiefly be concerned with the form of an inequality, rather than trying to find optimal constants, for example. To save on notation we will often denote by "C" or "c" a positive constant that does not depend on any other variables in the expression, unless specified otherwise. In the case that a constant does depend on some important variables, we may emphasise this using a parameter list, for example $C(\alpha, \beta, \ldots)$. In any case, the value of constants denoted by C or c may be different in each line where they appear.

1.2.2 Domains

The spatial dimension will sometimes be denoted by d and, when it is, we will always assume that $d \geq 2$. We will often use the notation $\Omega \subseteq \mathbb{R}^d$ for a simply-connected open set, which will usually be bounded or the whole space. We say that Ω is a *smooth domain* (or a C^k domain) if the boundary $\partial\Omega$ is smooth (or C^k), in the sense that $\partial\Omega$ is locally, at each point, the graph of a smooth (or C^k) function on a subset of $\mathbb{R}^{d-1}$, with respect to a suitable coordinate frame.

When considering functions satisfying a periodic boundary condition we will denote the (flat) d-torus by $\mathbb{T}^d \coloneqq \mathbb{R}^d/2\pi\mathbb{Z}^d$.

1.2.3 Continuous and smooth functions

For any $\Omega \subset \mathbb{R}^d$ (not necessarily open) we denote by $C(\Omega)$ the set of continuous functions on Ω and by $C_c(\Omega)$ the subspace consisting of those continuous functions with compact support. In certain situations we may simplify notation by omitting the domain, where the choice is clear from the context; in which case we may denote $C(\Omega)$ by

$$C^0 = C(\Omega)$$

since C alone may be confusing.

More generally, for $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we will denote by $C^m(\Omega)$ the space of functions on Ω with continuous derivatives up to order m, and by $C_c^m(\Omega)$ the space of those with compact support. Of course C^∞ and C_c^∞ will denote the spaces of functions for which all derivatives exist and, in the latter case, that also have compact support. Note that if Ω is compact then all continuous functions on Ω have compact support.

Now the set of bounded continuous functions on Ω forms a normed vector space with the supremum norm, which we denote by

$$\|f\|_\infty := \sup_{x \in \Omega} |f(x)|.$$

In addition, we say that $f : \Omega \to \mathbb{R}$ satisfies the γ-Hölder condition (or "f is γ-Hölder") for some $\gamma \in (0, 1]$ if there exists $C > 0$ such that

$$|f(x) - f(y)| < C|x - y|^\gamma \tag{1.1}$$

for all $x, y \in \Omega$. We then define spaces of γ-Hölder continuous functions by:

$$C^{0,\gamma}(\Omega) := \{f : \Omega \to \mathbb{R} : f \text{ satisfies a } \gamma\text{-Hölder condition for some } C > 0\}.$$

In particular $C^{0,1}$ is the space of Lipschitz functions.

We define a seminorm on $C^{0,\gamma}$ to encapsulate the γ-Hölder property (1.1):

$$\|f\|_{\dot{C}^{0,\gamma}} := \inf\{C > 0 : f \text{ is } \gamma\text{-Hölder, with coefficient } C\}.$$

Now bounded Hölder-continuous functions form a normed vector space with the norm

$$\|f\|_{C^{0,\gamma}} := \|f\|_\infty + \|f\|_{\dot{C}^{0,\gamma}}.$$

More generally, if $f \in C^{0,\gamma}$ has γ-Hölder derivatives up to order m, i.e. $D^\alpha f \in C^{0,\gamma}$ if $|\alpha| \leq m$, then we say $f \in C^{m,\gamma}(\Omega)$. In the case that f is bounded we also define a norm

$$\|f\|_{C^{m,\gamma}} := \|f\|_\infty + \sum_{0\leq|\alpha|\leq m} \|D^\alpha f\|_{\dot{C}^{0,\gamma}}.$$

Since any Hölder-continuous function is continuous we have, of course, that $C^{m,\gamma}(\Omega) \subset C^m(\Omega)$.

Note that when we discuss vector-valued functions $f : \Omega \to \mathbb{R}^d$, statements like "$f \in X$", for a normed space X, should be understood in a componentwise sense. In this case, the norm $\|\cdot\|_X$ should be understood as a norm on $|f|$:

$$\|f\|_X := \||f|\|_X.$$

1.2.4 Lebesgue Spaces

Much of the analysis in this work will concern functions in certain Lebesgue spaces or Sobolev spaces based upon them. In this subsection we will set out the notation we will use when working with these spaces and recall a few standard facts. More detailed discussion can be found in countless textbooks, for example Adams (1975), Robinson (2001) or Cohn (1980).

Unless otherwise specified, all integrals over subsets of $\mathbb{R}^d$ will be written with respect to the Lebesgue measure λ in the corresponding dimension. For $1 \leq p < \infty$ and any λ-measurable set $\Omega \subset \mathbb{R}^d$ the space $L^p(\Omega)$ (which will usually be denoted by L^p, when the choice of domain is clear) consists of all equivalence classes of functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f(x)|^p \,\mathrm{d}x < \infty,$$

where f is equivalent to another function g if $\lambda\{x \in \Omega : f \neq g\} = 0$. For the endpoint $p = \infty$, we define L^∞ to be the space of equivalence classes of functions that are essentially bounded:

$$\operatorname{ess\,sup}_\Omega |f| \coloneqq \inf\{\sup\{|f(x)| : x \in \Omega\backslash E\} : \lambda(E) = 0\} < \infty.$$

That is, there exists a λ-null set E such that $\sup_{\Omega\backslash E} |f| < \infty$.

As always we will treat L^p as a space of functions rather than a quotient space, for example notation of the form $f \in L^p$ will be abused frequently, to denote that "f is a representative of a class in $L^p(\Omega)$" for $1 \le p \le \infty$.

The set L^p forms a linear space under the pointwise addition of functions and moreover is a Banach space (a complete normed space) with the norm

$$\|f\|_{L^p(\Omega)} \coloneqq \left(\int_\Omega |f|^p\right)^{1/p}$$

if $1 \le p < \infty$, or

$$\|f\|_{L^\infty(\Omega)} \coloneqq \operatorname{ess\,sup}_\Omega |f|$$

for $p = \infty$.

If $\Omega \subseteq \mathbb{R}^d$ is unbounded, we will sometimes consider functions that are only locally in L^p, in the sense that for any compact subset $K \Subset \Omega$ they may be restricted to an $L^p(K)$ function, but do not have sufficient "decay" to be in $L^p(\Omega)$. We denote the space of such functions by $L^p_{\text{loc}}(\Omega)$.

One of the key families of inequalities that we have for estimating integrals using L^p norms are the Hölder inequalities, which we will use very frequently.

Lemma 1.1. *For any $1 \le p, q \le \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{1.2}$$

if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then fg is integrable and

$$\left|\int_\Omega fg \,\mathrm{d}x\right| \le \|f\|_{L^p}\|g\|_{L^q}.$$

Note that p, q satisfying (1.2) are called *conjugate exponents.* The following corollary gives a useful generalisation.

Corollary 1.2. *If $f_1, \dots, f_n$ are functions such that $f_i \in L^{p_i}$ for $1 \le i \le n$*

and

$$\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{r}$$

for some $r \geq 1$*, then the product* $f_1 f_2 \ldots f_n \in L^r$ *and*

$$\left\| \prod_{i=1}^{n} f_i \right\|_{L^r} \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}}.$$

A useful consequence of the Hölder inequalities is Lebesgue interpolation.

Corollary 1.3 (Lebesgue interpolation)**.** *Let* $1 \leq p < q < r \leq \infty$*, if* $f \in L^p \cap L^r$ *then* $f \in L^q$ *and*

$$\|f\|_{L^q} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^r}^{1-\theta}$$

where θ *satisfies*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

Another important, yet elementary, family of inequalities that we will use repeatedly throughout this work follow from Young's inequality for products. If f, g, p and q satisfy the same hypotheses as in Lemma 1.1 then

$$\int_\Omega fg \leq \frac{\varepsilon^p}{p} \|f\|_{L^p}^p + \frac{1}{\varepsilon^q q} \|g\|_{L^q}^q$$

for any $\varepsilon > 0$.

For $1 \leq p < \infty$, the dual space of $L^p(\Omega)$ is isometrically isomorphic to $L^q(\Omega)$ if p and q are conjugate exponents. In particular the dual of L^1 can be identified with L^∞ (but not vice versa). More precisely, for any bounded linear functional T on L^p there exists a unique $g \in L^q$ such that

$$\|T\|_{(L^p)^*} = \|g\|_{L^q}$$

and for any $f \in L^p$

$$\langle T, f \rangle_{(L^p)^* \times L^p} = \int_\Omega fg \, \mathrm{d}x. \tag{1.3}$$

We will use the notation

$$\langle T, \cdot \rangle_{X^* \times X} = T(\cdot)$$

where X is a normed space and $T \in X^*$ is a bounded linear functional. However, in the case that a functional is given by a function g as described above, we may instead use the notation

$$\langle g, \cdot \rangle_{X^* \times X} = T_g(\cdot),$$

where T_g is the functional corresponding to g, as in (1.3)

In the case that $p = 2$, L^2 is a Hilbert space with the inner product

$$(f, g)_{L^2} := \int_\Omega f g \,\mathrm{d}x,$$

or

$$(u, v)_{L^2} := \sum_i (u_i, v_i)_{L^2},$$

in the case of vector valued functions u and v.

Another important property of $L^p(\Omega)$ is seperability for $1 \leq p < \infty$ and for any domain $\Omega \subset \mathbb{R}^d$ or $\Omega \subset \mathbb{T}^d$. Moreover smooth compactly supported functions $C_c^\infty(\Omega)$ are dense in L^p for $1 \leq p < \infty$.

1.2.5 Derivatives

For partial derivatives of sufficiently smooth functions (scalar or vector valued) we will often use the shorthand

$$\partial_i := \partial_{x_i} := \frac{\partial}{\partial x_i}$$

for the partial derivative in the ith principle spatial direction. Similarly

$$\partial_t := \frac{\partial}{\partial t}$$

denotes the time derivative. We will sometimes use the multi-index notation for multiple partial derivatives, i.e. if $\alpha \in \mathbb{N}_0^d$, then

$$D^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_d^{\alpha_d}.$$

The above notation will be also be used for weak spatial derivatives or sometimes for weak derivatives in the sense of Bochner spaces (see below).

For a function $f \in L^1_{\mathrm{loc}}(\Omega)$ we say that $g \in L^1_{\mathrm{loc}}(\Omega)$ is a *weak derivative* of f (or $g = \partial_i f$, in a suitable context) if

$$\int_\Omega g\phi \,\mathrm{d}x = -\int_\Omega f \partial_i \phi \,\mathrm{d}x$$

for all test functions $\phi \in \mathrm{C}_c^\infty(\Omega)$. We define multiple derivatives inductively, in the obvious way, as follows. For $m \in \mathbb{Z}$ with $m > 0$, we say f has $m+1$ weak derivatives (with respect to x_i) if it has m weak derivatives and $\partial_i^m f$ is weakly differentiable:

$$\partial_i^{m+1} f := \partial_i \partial_i^m f.$$

It follows easily from the definition that weak derivatives commute, so more generally multiple weak derivatives

$$D^\alpha f$$

are well defined, if they exist.

1.2.6 Fourier Transforms

The Fourier basis for $L^2(\mathbb{T}^d)$ consists of periodic functions of the form

$$x \mapsto \frac{1}{(2\pi)^{d/2}} \mathrm{e}^{\mathrm{i}x\cdot k}$$

where $k \in \mathbb{Z}^d$, and the Fourier coefficients of a function $f \in L^2(\mathbb{T}^d)$ will be denoted by $\hat{f}(k) \in \mathbb{C}$ (or $\hat{f}(k) \in \mathbb{C}^d$ if f is vector valued). The formula that defines $\hat{f}(k)$ is

$$\hat{f}(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(x) \mathrm{e}^{-\mathrm{i}k\cdot x} \,\mathrm{d}x,$$

and the corresponding decomposition of f is

$$f(x) = \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} \hat{f}(k) \mathrm{e}^{\mathrm{i}x\cdot k}.$$

Note that if all components of f are real valued then $\hat{f}(k) = \overline{\hat{f}(-k)}$ for all $k \in \mathbb{Z}^d$ where $\overline{x}$ denotes the complex conjugate.

Occasionally we will denote by $P_n : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ the projection onto Fourier modes of order at most $n \geq 0$ i.e.

$$P_n f = \frac{1}{(2\pi)^{d/2}} \sum_{|k|\leq n} \hat{f}(k) e^{\mathrm{i}x\cdot k}.$$

In particular, for brevity we may express the fact that the Fourier coefficients of a function f vanish above order n, by saying that $f = P_n f$.

1.2.7 Sobolev spaces

A significant amount of the analysis in this work will be carried out in Sobolev spaces. Loosely speaking these are spaces of L^p functions with weak derivatives in L^p. More precisely, for $m \in \mathbb{N}_0$ and $1 \leq p < \infty$, the space $W^{m,p}$ consists of functions $u \in L^p$ such that the weak derivatives $D^\alpha u$ exist and are in L^p for all multi-indices α such that $|\alpha| \leq m$. On this space we define the Sobolev norm

$$\|u\|_{W^{m,p}} := \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p}^p \right)^{1/p}.$$

In the case that $p = 2$, we use the notation $H^m := W^{m,2}$ since this is a Hilbert space with the inner product

$$(f, g)_{H^m} := \sum_{|\alpha|\leq m} (D^\alpha f, D^\alpha g)_{L^2}.$$

These spaces should not be confused with Hardy spaces, which are often denoted in the same way but will not be discussed in this thesis.

We will use the notation $W_0^{m,p}(\Omega)$ (or $H_0^m(\Omega)$) for the closure of $C_c^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{m,p}}$ (or $\|\cdot\|_{H^m(\Omega)}$).

For the dual spaces of $W_0^{m,p}$ (or H_0^m) we will use the notation $W^{-m,p}$ (or H^{-m}) respectively. As with the Lebesgue spaces, it will be important for us to be able to identify functions that give rise to a bounded functional on $W_0^{m,p}$, in the usual way. For $f \in L^1_{\mathrm{loc}}(\Omega)$, we may abuse the notation $f \in W^{-m,p}$ to mean that for any $g \in W_0^{m,p}$, gf is integrable and

$$\left| \int_\Omega g f \, \mathrm{d}x \right| \leq C \|g\|_{W_0^{m,p}}.$$

As described in the above comments about Lebesgue spaces, we may also denote the integral on the left-hand side by

$$\langle f, g\rangle_{W^{-m,p}\times W_0^{m,p}} = \int_\Omega g f \,\mathrm{d}x.$$

We now give the statements of several important standard theorems regarding these spaces, which will be used throughout the rest of this work. The proofs and further information can be found in Adams (1975) or Leoni (2009).

We begin with a theorem that is essentially due to Meyers and Serrin (1964), that gives the density of smooth functions in $W^{m,p}$. This result can be generalised for less regular domains, but is sufficient for our purposes.

Theorem 1.4. *If $\Omega \subseteq \mathbb{R}^d$ is a smooth (but not necessarily bounded) domain then $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ for $m \in \mathbb{N}_0$ and $1 \le p < \infty$. Moreover, the restrictions of functions in $C_c^\infty(\mathbb{R}^d)$ onto Ω are dense in these spaces. In particular $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$, so $W^{m,p}(\mathbb{R}^d) = W_0^{m,p}(\mathbb{R}^d)$.*

Next we have several important Sobolev embedding results. We say that a normed space $X(\Omega)$ of (equivalence classes of) functions on Ω is continuously embedded in another, $Y(\Omega)$, if $X \subset Y$ and the inclusion is linear with

$$\|f\|_Y \le C\|f\|_X$$

for some $C > 0$ independent of f. We denote this by $X \hookrightarrow Y$.

Theorem 1.5. *Let $\Omega \subseteq \mathbb{R}^d$ be a smooth domain (not necessarily bounded).*
Case 1: *If $mp < d$ then*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

if $p \le q \le p^*$, where

$$p^* = \frac{dp}{d - mp}.$$

Moreover, for any $j \in \mathbb{N}$,

$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega).$$

Case 2: *If $mp = d$ then*

$$W^{m,p}(\Omega) \hookrightarrow L^q$$

if $p \leq q < \infty$.
Case 3: *If* $mp > d + k > (m-1)p$, *for some* $k \in \mathbb{N}_0$, *then*

$$W^{m,p}(\Omega) \hookrightarrow C^{k,\lambda}(\Omega),$$

if $0 < \lambda \leq (mp - d - k)/p$. *In the sense that functions* $W^{m,p} \subset C^{k,\lambda} \cap L^\infty$ *and*

$$\|f\|_{C^{k,\lambda}} \leq C\|f\|_{W^{m,p}}$$

for some $C > 0$, *for all* $f \in W^{m,p}$.

We say a domain Ω is *bounded in one direction* if for some $v \in \mathbb{R}^d \backslash \{0\}$

$$x \in \Omega \Rightarrow |x \cdot v| \leq C$$

for some $C > 0$. For such domains we have the important Poincaré inequality (see, for example, Chapter 5 of Evans (2010) and Chapter 12 of Leoni (2009)).

Theorem 1.6. *If* Ω *is bounded in one direction and* $1 \leq p < \infty$ *there exists a constant* $C > 0$ *such that*

$$\|f\|_{L^p} \leq C\|\nabla f\|_{L^p}$$

for any $f \in W^{1,p}(\Omega)$ *such that either* $f \in W_0^{1,p}$ *or*

$$\int_\Omega f \, \mathrm{d}x = 0.$$

On bounded domains, some of the embeddings in Case 1 of Theorem 1.5 are in fact compact. We say that an embedding $E : X \hookrightarrow Y$ is compact if the image $E(Z)$ is totally bounded (relatively compact) for any bounded subset $Z \subset X$. In particular if $Z = \{z_n\}$ is a sequence, then $E(Z)$ admits a subsequence converging in Y. We denote a compact embedding by $X \Subset Y$.

Theorem 1.7. *If* Ω *is a smooth bounded domain then the embedding*

$$W^{m,p}(\Omega) \hookrightarrow L^q$$

is compact if $mp < d$ *and* $1 \leq q < p^*$ *or* $mp = d$ *and* $1 \leq q < \infty$.

Another important property of higher order Sobolev spaces i.e. $W^{m,p}$ for $mp > p^*$ is that these spaces are Banach algebras, as described in the following theorem.

Theorem 1.8. *If $m \in \mathbb{N}_0$ and $1 \leq p < \infty$ such that $mp > p^*$ and Ω is a smooth domain (not necessarily bounded) then for any $f, g \in W^{m,p}(\Omega)$, the product $fg \in W^{m,p}(\Omega)$ and there exists $C > 0$ independent of f and g such that*

$$\|fg\|_{W^{m,p}} \leq C\|f\|_{W^{m,p}}\|g\|_{W^{m,p}}.$$

For (non-integer) $s \geq 0$, we use the following definition of the (inhomogeneous) Sobolev space $H^s(\mathbb{T}^3)$. For $f \in L^2(\mathbb{T}^3)$, we say $f \in H^s(\mathbb{T}^3)$ if the Fourier coefficients satisfy

$$\sum_{k\in\mathbb{Z}^3} |k|^{2s}|\hat{f}(k)|^2 < \infty.$$

For $f \in H^s(\mathbb{T}^3)$, we define the "modulus of s derivatives" Λ^s by

$$\Lambda^s f(x) := (2\pi)^{-3/2} \sum_{k\in\mathbb{Z}^3} |k|^s \hat{f}(k) \mathrm{e}^{\mathrm{i}k\cdot x} \in L^2(\mathbb{T}^3).$$

In particular $\Lambda^2 f = (-\Delta)f$ for any $f \in H^2$.

Moreover we will denote by $\|\cdot\|_s$ the seminorm $\|\Lambda^s \cdot\|_{L^2}$. Using this notation, the norm in H^s is given by

$$\|\cdot\|_{H^s} := \left(\|\cdot\|_{L^2}^2 + \|\cdot\|_s^2\right)^{1/2}.$$

Note that we will sometimes use the fact that this is equivalent to the norm $\|\cdot\|_{L^2} + \|\cdot\|_s$. We will also make use of the fact that for a function $f \in H^t(\mathbb{T}^3)$, $\|f\|_s \leq \|f\|_t$ if $0 < s \leq t$.

Almost analogously, one can define $H^s(\mathbb{R}^3)$, using Fourier transforms (see for example Bahouri, Chemin, and Danchin (2011)).

1.2.8 Bochner-Sobolev spaces

In this thesis, following the typical approach to parabolic PDEs, we will usually think of (weak) solutions as functions of time with values in certain Banach spaces. The (weak) time derivative is then understood as a weak derivative in the sense of Bochner integrals, i.e. for a Banach space X, we say $\partial_t u = v \in L^1_{\text{loc}}(0,T;X)$ if $u \in L^1_{\text{loc}}(0,T;X)$, and v satisfies the following identity of Bochner integrals:

$$\int_0^T u(s)\partial_t \phi(s)\ \mathrm{d}s = -\int_0^T v(s)\phi(s)\ \mathrm{d}s \in X,$$

for any $\phi \in C_c^\infty(0,T)$.

Following Chapter 7 of Roubíček (2013), we will use the following notation

$$W^{1,p,q}(0,T;X,Y) := \{f \in L^p(0,T;X) : \partial_t f \in L^q(0,T;Y)\}.$$

This space is equipped with the norm

$$\|f\|_{W^{1,p,q}(0,T;X,Y)} := \|f\|_{L^p(0,T;X)} + \|\partial_t f\|_{L^q(0,T;Y)}.$$

Most of the existence results in this thesis use the following compactness Lemma, originally due to Aubin (1963) and Lions (1969) (see also the necessary and sufficient conditions for sets to be relatively compact in $L^p(0,T;X)$ due to Simon (1987)). We will use a slightly weaker version of the result Roubíček proves. Other versions can be found; see for example the compactness argument in Chapter 8 of Constantin and Foias (1988).

Lemma 1.9 (Aubin-Lions)**.** *Let X, Y, Z be Banach spaces with X also separable and reflexive. If we have the continuous embedding $Y \hookrightarrow Z$ and the compact embedding $X \Subset Y$, then for $p \in (1,\infty)$ and $q \in [1,\infty]$ the following compact embedding holds:*

$$W^{1,p,q}(0,T;X,Z) \Subset L^p(0,T;Y).$$

These embeddings can be extended to higher-order Bochner-Sobolev spaces, i.e. we can consider more derivatives in time. Indeed we have the following corollary in the case $p = q$

Corollary 1.10. *Let X, Y, Z be spaces as in Lemma 1.9 with Z also reflexive, and fix $p \in (1,\infty)$. Suppose $(u_n)_{n=1}^\infty$ and $(\partial_t^k u_n)_{n=1}^\infty$ are bounded sequences in*

$$W^{1,p,p}(0,T;X,Z),$$

for some range of $k = 1,2,\dots,K$. Then there exists $u \in L^p(0,T;Y)$ with

K (weak) time derivatives in $L^p(0,T;Y)$, such that $u_n \to u$ in $L^p(0,T;Y)$, and for each $k = 1, \dots, K$

$$\partial_t^k u_n \to \partial_t^k u.$$

Proof. After applying Lemma 1.9 to each derivative, it suffices to check that the limits are consistent, i.e. that $\partial_t^k u_n \to \partial_t^k u$. This can be proved inductively. For example, from the first application of the lemma we may assume that $\partial_t u_n$ converges weakly to $\partial_t u$ in $L^p(0,T;Z)$ (since this space is reflexive). In the second application we see that $\partial_t u_n$ converges in $L^p(0,T;Y)$. Since $Y \hookrightarrow Z$ is a continuous embedding, the limits must agree. □

The following two lemmas will also be useful. The proofs can, again, be found in Chapter 7 of Roubíček (2013) (see also Exercise 6.1 in Robinson, Rodrigo, and Sadowski (2016)). They will be used when proving estimates on weak solutions to certain PDEs, by effectively allowing us to use much weaker "test functions". In particular, we will deduce Corollary 1.14 and consequently Lemma 1.15 which gives density of C_c^∞ in $W^{1,2,2}(0,T;H_0^1,H^{-1})$ in a sufficiently strong sense for our purposes.

Lemma 1.11. *Let $p, q \geq 1$ and X, Y Banach spaces[2] such that there is a continuous embedding $X \hookrightarrow Y$, then for any $T > 0$, $C^1([0,T];X)$ is dense[3] in $W^{1,p,q}(0,T;X,Y)$.*

The proof is essentially a careful mollification argument using Bochner integrals. Given $u \in W^{1,p,q}(0,T;X,Y)$ the $C^1([0,T];X)$ approximations constructed by Roubíček are of the form

$$u_\varepsilon(t) = \int_0^T \rho_\varepsilon(t,s)u(s)\ \mathrm{d}s,$$

where for all $t \in [0,T]$, and $\varepsilon > 0$

$$\int_0^T \rho_\varepsilon(t,s)\ \mathrm{d}s = 1.$$

Note that Roubíček chooses ρ_ε to depend on t in such a way that $\rho_\varepsilon(t,\cdot)$ is always supported on $[0,T]$, rather than being a function of $t-s$ alone.

[2] Actually, Roubíček shows that this still holds if Y is locally convex, rather than a Banach space.

[3] Here C^1 means functions with a continuous Fréchet derivative.

It is not difficult to see that the mollification adds more derivatives than stated. In fact one can check that $u_\varepsilon \in C^\infty([0,T];X)$.

Using Lemma 1.11, Roubíček also proves the following embedding and identity.

Lemma 1.12. *Let X, Y be a separable Banach space and a separable Hilbert space, respectively, such that $X \hookrightarrow Y \cong Y^* \hookrightarrow X^*$ and let p, $p' > 1$ be conjugate exponents. Then there is a continuous embedding*

$$W^{1,p,p'}(0,T;X,X^*) \subset C([0,T];Y).$$

Moreover, we have the integration by parts formula

$$(f(t),g(t))_Y - (f(s),g(s))_Y = \int_s^t \left\langle \frac{\mathrm{d}f}{\mathrm{d}t}, g(r) \right\rangle_{X^* \times X} + \left\langle \frac{\mathrm{d}g}{\mathrm{d}t}, f(r) \right\rangle_{X^* \times X} \mathrm{d}r \tag{1.4}$$

for any $0 \le s \le t \le T$.

It is important to note that if $u \in W^{1,p,p'}(0,T;X,X^*)$ is weakly continuous in Y, at a time t, i.e. $(u(r),v)_Y$ is continuous at $r = t$ for any $v \in Y$, then we need not modify u at t to find the continuous representative.

In this thesis, we usually estimate linear functionals (elements of H^{-1}, for example) using the identification $L^2 \cong (L^2)^*$. In order to use Lemma 1.12 to obtain continuous representatives in higher-order Sobolev spaces we observe that (weak) derivatives in time commute with those in space, i.e. for $u \in W^{1,p,p'}(0,T;H^{1+k},H^{1-k})$

$$\partial_x^k \partial_t u = \partial_t \partial_x^k u$$

for any spatial derivative ∂_x^k of order k. In the context of the torus $\mathbb{T}^d$, we see that Λ^s commutes with ∂_t in the same sense. The following corollary can be proved on the whole space or $\mathbb{T}^d$, using this observation and the previous lemma. In the case of a bounded domain, see Section 5.9 of Evans (2010).

Corollary 1.13. *For Ω a smooth bounded domain, $\mathbb{T}^d$, or $\mathbb{R}^d$, the following continuous embeddings hold:*

$$W^{1,2,2}(0,T;H^{s+1}(\Omega),H^{s-1}(\Omega)) \hookrightarrow C([0,T];H^s(\Omega))$$

for any $T > 0$ and any $s \geq 0$ (here $s \in \mathbb{N}_0$ unless the domain is $\mathbb{T}^d$).

Another consequence of Lemmas 1.11 and 1.12 is the following corollary. The proof is essentially a part of the proof of Lemma 1.12 that Roubíček left implicit. See also[4] Theorem 7.2 of Robinson (2001).

Corollary 1.14. *Let X, Y, p and p' be as in the statement of Lemma 1.12. Then for any $T > 0$ and any $u \in W^{1,p,p'}(0,T;X,X^*)$ there exists a sequence $(\phi_n)_{n=1}^{\infty} \subset C^{\infty}([0,T];X)$ such that $\phi_n \to u$ in $W^{1,p,p'}(0,T;X,X^*)$ and $\phi_n \to u$ in $C([0,T];Y)$ (uniform convergence).*

Proof. Using Lemma 1.11, let us take a sequence $(\phi_n)_{n=1}^{\infty}$ in $C^{\infty}([0,T];X)$ that converges to u in $W^{1,p,p'}(0,T;X,X^*)$.

To prove that ϕ_n converges uniformly in Y, we essentially follow estimates from Roubíček's proof of Lemma 1.12. We may assume that $\phi_n(t) \to u(t)$ in Y for almost every $t \in [0,T]$. Indeed, ϕ_n converges in $L^p(0,T;X) \subset L^p(0,T;Y)$.

We will assume that (1.4) holds for the approximations ϕ_n. For simplicity we will only treat the case $p = p' = 2$, since the generalisation is not difficult (see Roubicek's proof of Lemma 1.12).

Fix any $n, m \in \mathbb{N}$ and $t \in (0,T]$ and let $s = s^* \in [0,T]$ be such that

$$\|\phi_n(s^*) - \phi_m(s^*)\|_Y^2 = \frac{1}{T}\int_0^T \|\phi_n(\tau) - \phi_m(\tau)\|_Y^2 \,\mathrm{d}\tau,$$

which exists by continuity of ϕ_n and ϕ_m. By (1.4) we see that

$$\begin{aligned}
&\|\phi_n(t) - \phi_m(t)\|_Y^2 \\
&\leq \frac{1}{T}\|\phi_n - \phi_m\|_{L^2(0,T;Y)}^2 + 2\left\|\frac{\mathrm{d}}{\mathrm{d}t}(\phi_n - \phi_m)\right\|_{L^2(0,T;X^*)} \|\phi_n - \phi_m\|_{L^2(0,T;X)}
\end{aligned} \tag{1.5}$$

where we have switched the roles of s and t if $s^* > t$. Since the right-hand side is independent of t, we see that ϕ_n is uniformly Cauchy in $C([0,T];Y)$.

We can conclude that, up to an adjustment on a set of measure zero, ϕ_n converges to u in $C([0,T];Y)$ uniformly. □

[4]The mollification argument used to prove Theorem 7.2 of Robinson (2001) appears to have a flaw, but we can safely proceed analogously with the argument that follows it.

In practice we will usually take $X = H_0^1$ and $Y = L^2$. In various uniqueness arguments in later chapters we will need $\phi_n \in C_c^\infty([0,T) \times \Omega)$, to converge in $W^{1,p,p'}(0,S;X,X^*)$ and in $C([0,S];Y)$ to a given function $u \in W^{1,p,p'}(0,T;X,X^*)$ and given $S \in (0,T)$. The following lemma is a sufficient extension of Corollary 1.14 for these purposes. The proof is partly based on the proof of Lemma 3.11 from Robinson et al. (2016).

Lemma 1.15. *Let $\Omega = \mathbb{R}^d$, $\mathbb{T}^d$ or a smooth bounded domain in $\mathbb{R}^d$. For any $u \in W^{1,2,2}(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ and any $S \in (0,T)$ there exists a sequence $(\phi_n)_{n=1}^\infty \subset C_c^\infty([0,T) \times \Omega)$, that converges to u in $W^{1,2,2}(0,S;H_0^1,H^{-1})$ and in $C([0,S];L^2)$.*

Proof. Let ϕ_n be the sequence of approximations constructed in Corollary 1.14 on a time interval $[0,S']$ for some $S' \in (S,T)$. After multiplying by a smooth cutoff function $\psi(t)$ such that $\chi_{[0,S]} \le \psi \le \chi_{[0,S']}$, we may assume that $\phi_n \in C_c^\infty([0,T);H_0^1(\Omega))$ with the required convergence properties on $[0,S]$.

In the case $\Omega = \mathbb{R}^d$ we smoothly truncate the functions ϕ_n in space. Indeed, we can consider compactly supported approximations to ϕ_n of the form

$$\xi_\ell^n = K_\ell \phi_n,$$

Here $K_\ell \in C_c^\infty$ is a smooth cutoff function supported on $B_{\ell+1}(0)$ and identically 1 on $B_\ell(0)$, with derivatives bounded uniformly, that is

$$\|\nabla K_\ell\|_{L^\infty} \le C$$

for a constant C, independent of ℓ. It is not difficult to check that ξ_ℓ^n converges to ϕ_n in $C^k([0,T];H^1(\mathbb{R}^d))$ if $\phi_n \in C^k([0,T];H^1(\mathbb{R}^d))$, hence also $\xi_\ell^n \to \phi_n$ in $W^{1,p,p'}(0,T;H^1,H^{-1})$

It now suffices to consider the cases where Ω is a bounded domain or $\Omega = \mathbb{T}^d$. Fix an orthonormal basis $(a_i)_{i=1}^\infty \subset C^\infty(\Omega)$ of H_0^1 and denote by Π_N the projection of H_0^1 onto $\{a_i : 1 \le i \le N\}$. Since $\phi_n \in C_c^\infty([0,T);H_0^1)$ we have

$$\Pi_N \phi_n(x,t) = \sum_{i \le N} \beta_i(t) a_i(x)$$

where $\beta_i \in C_c^\infty([0,T))$ for each i. It is easy to check that $\Pi_N \phi_n(t) \to \phi_n(t)$ in H_0^1, uniformly on $[0,S]$ with respect to t, as $N \to \infty$. Likewise, taking

k time derivatives, we have

$$\partial_t^k \Pi_N(\phi_n(t)) = \Pi_N(\partial_t^k \phi_n(t)) \to \partial_t^k \phi_n(t)$$

in H_0^1, uniformly with respect to t as $N \to \infty$. Indeed,

$$\left\| \Pi_N(\partial_t^k \phi_n(t)) - \partial_t^k \phi_n(t) \right\|_{H_0^1}$$

is continuous with respect to t for each N, and decreases to 0 as $N \to \infty$ for any fixed t. Hence convergence is uniform on $[0, S]$, by Dini's lemma (see Rudin (1976), Theorem 7.13).

That $\Pi_N \phi_n$ converges to ϕ_n in $W^{1,p,p'}(0, S; H_0^1, H^{-1})$ and $C([0, S]; L^2)$ follows from the convergence in $C^\infty([0, S]; H_0^1)$. It remains to find a sequence in $C_c^\infty([0, T) \times \Omega)$ that converges to $\Pi_N \phi_n$ in $C^\infty([0, S]; H_0^1)$.

In the case $\Omega = \mathbb{T}^3$, we already assumed that $a_i \in C_c^\infty(\Omega)$ so it suffices to consider the case of a smooth bounded domain Ω. In that case, for each $i \geq 1$, we may consider a sequence of compactly supported functions $(\alpha_i^k)_{k=1}^\infty \subset C_c^\infty(\Omega)$ such that $\alpha_i^k \to a_i$ in H_0^1. It is easy to check that

$$\sum_{i \leq N} \beta_i \alpha_i^k \to \Pi_N \phi_n$$

in $C^\infty([0, S]; H_0^1)$, as $k \to \infty$, and

$$\sum_{i \leq N} \beta_i \alpha_i^k \in C_c^\infty([0, T) \times \Omega),$$

as required. □

1.2.9 The Helmholtz-Weyl decomposition and the Leray projector

A well-known family of results, most commonly attributed to Helmholtz (1858, 1870) (see also an earlier result by Stokes (1856)), show that a smooth vectorfield on $\mathbb{R}^3$ with sufficiently fast decay (or compact support) can be decomposed into a divergence-free part, and a curl-free (gradient) part:

$$u = \nabla \times h + \nabla g.$$

Much later this observation was extended by Weyl and others, to prove a decomposition of the Lebesgue spaces L^p, $1 < p < \infty$. For a discussion of such results on various domains, see for example Galdi (2011) or Robinson et al. (2016). For our purposes it will suffice to consider the cases of $L^2(\mathbb{T}^d)$, and $L^2(\mathbb{R}^d)$, for $d \geq 2$. In either domain we have

$$L^2 = H \oplus G,$$

where H is the closure of the set of smooth divergence-free functions in L^2, and G is the space of gradients of H^1 functions. By considering Fourier series, this decomposition can be written explicitly (see, for example Chapter 2 of Robinson et al. (2016)). Indeed for $u \in L^2(\mathbb{T}^d)$,

$$u(x) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} \hat{u}(k) e^{\mathrm{i}x \cdot k} = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} (\hat{g}(k) + \hat{h}(k)) e^{\mathrm{i}x \cdot k}$$

where

$$\hat{g}(k) \coloneqq \frac{\hat{u}(k) \cdot k}{|k|^2} k, \text{ for } k \neq 0,$$

$\hat{g}(0) \coloneqq 0$, and $\hat{h}(k) \coloneqq \hat{u}(k) - \hat{g}(k)$. It is straightforward to check that $\hat{g}$ and $\hat{h}$ are the coefficients of convergent Fourier series, let us call the corresponding limits g and h, respectively. It is also not difficult to see that g is the weak derivative of the scalar-valued H^1 function f, with Fourier coefficients

$$\hat{f}(k) = -\mathrm{i} \frac{\hat{u}(k) \cdot k}{|k|^2}, \ \hat{f}(0) = 0.$$

Moreover, it can be seen that $h \in H$ since $\hat{h}(k) \cdot k = 0$ for all $k \in \mathbb{Z}^d$. A similar approach can be taken for the case of $L^2(\mathbb{R}^d)$.

To see that the decomposition of a given function u is unique, it suffices to consider $u = 0$. In that case $h = -g = \nabla f$, in a weak sense for some $f \in H^1$. Formal consideration of the Fourier series of f, assuming that $\nabla \cdot h = 0$, implies that $\hat{f}(k)|k|^2 = 0$ for all $k \in \mathbb{Z}^d$, hence $h = g = 0$. This can be justified by considering the Fourier series of a sequence of smooth divergence-free approximations to h.

The projection of L^2 onto H will play an important role in the analysis herein; we will denote it by

$$\mathbb{P} : L^2 \to H. \tag{1.6}$$

On $L^2(\mathbb{R}^d)$ or $L^2(\mathbb{T}^d)$, $\mathbb{P}$ can be calculated explicitly in Fourier space, following the discussion above. For example, on $\mathbb{T}^d$ we have

$$\mathbb{P}\left(\sum_{k\in\mathbb{Z}^d} \hat{u}(k)\mathrm{e}^{\mathrm{i}x\cdot k}\right)(x) = \hat{u}(0) + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \left(\hat{u}(k) - \frac{\hat{u}(k)\cdot k}{|k|^2}k\right)\mathrm{e}^{\mathrm{i}x\cdot k}.$$

This is usually called the *Leray projection* (or sometimes the Helmholtz projection). Clearly $\mathbb{P}$ is a bounded operator on L^2, moreover it follows easily from the Fourier-series definition that for any $s \geq 0$ and any $u \in H^s(\mathbb{T}^d)$

$$\|\Lambda^s \mathbb{P} u\|_{L^2} \leq \|\Lambda^s u\|_{L^2}.$$

Furthermore $\mathbb{P}$ and Λ^s commute on $H^s(\mathbb{T}^d)$ (this is discussed in the aforementioned references).

Chapter 2

Fluid mechanics background

2.1 Introduction to the Euler and Navier–Stokes equations

The Euler equations were first described in print by Leonhard Euler (1755) in order to model the motion of an inviscid fluid; that is, one where the effects of internal friction are negligible. The classical incompressible Euler equations in a domain $\Omega \subseteq \mathbb{R}^d$ for $d \geq 2$ on a time interval $I \subseteq \mathbb{R}$ comprise the system

$$\partial_t u + (u \cdot \nabla)u + \nabla p = f \tag{2.1}$$

$$\nabla \cdot u = 0 \tag{2.2}$$

where $u \colon \Omega \times I \to \mathbb{R}^d$ is an evolving vectorfield representing the fluid velocity, $p \colon \Omega \times I \to \mathbb{R}$ is an evolving potential representing the pressure and $f \colon \Omega \times I \to \mathbb{R}^d$ is a given "body-force", which can be used to represent a phenomenon extrinsic to the fluid itself, for example gravity.

Unless stated otherwise, in this thesis when we refer to "the Euler equations" we mean the (incompressible) Euler equations as defined above, in the homogeneous case ($f = 0$). This is to keep the analysis as clear as possible. It is reasonable to expect that most of the results herein can be generalised to apply to the inhomogeneous problem, for sufficiently regular functions f.

Here and throughout this work we use the notation

$$(u \cdot \nabla)v = u_i \partial_i v$$

for a differentiable scalar-valued function v, where we sum over the repeated index i. If v is vector valued then this is applied componentwise. We have denoted the divergence by

$$\nabla \cdot u := \operatorname{div} u := \partial_i u_i.$$

The Navier–Stokes equations are so named to reflect the contributions of Navier (1822) and Stokes (1845). As an aside, we observe that in his papers Stokes acknowledges an alternative derivation by Poisson, contemporary with his own. The equations are obtained by adding a diffusion term to the Euler equations to model viscous effects (friction) within the fluid:

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f \tag{2.3}$$

$$\nabla \cdot u = 0 \tag{2.4}$$

where the coefficient $\nu > 0$ is called the *viscosity* and

$$\Delta u = \partial_i \partial_i u$$

denotes the Laplacian.

Defining our terms as we did for the Euler equations, in this thesis "the Navier–Stokes equations" will refer to the (incompressible) Navier–Stokes equations as defined above, in the homogeneous case. Moreover, the choice of ν does not have a qualitative effect on many of the estimates herein. We will therefore simplify our calculations by assuming that $\nu = 1$, unless stated otherwise. If further justification is needed, consider the fact that (u, p) is a solution of the Navier–Stokes equations on $\Omega \times [0, T)$ for $\nu > 0$ if and only if

$$v(x,t) := \nu^{-1} u\left(x, t/\nu\right), \; q(x,t) := \nu^{-2} p\left(x, t/\nu\right)$$

is a solution for $\nu = 1$ on $\Omega \times [0, \nu T)$.

In the remainder of this chapter we will discuss the derivation of the Navier–Stokes equations, which applies equally to the Euler equations, before giving very condensed reviews of certain parts of the known theory for each system. There is a vast amount of literature available, given the history of these two systems, so we aim only to highlight selected results, problems and references in order to set the results of later chapters in the proper context.

2.2 Derivation of the equations

In this section we give a derivation of the Navier–Stokes equations for incompressible flow in 3D (from which the Euler equations can be derived). The analogous equations in d-dimensions for $d \geq 2$ can be derived similarly. This is not meant to be a complete explanation of the required continuum mechanics, although we will attempt to briefly justify the arguments from the first principles of Newtonian mechanics. Our discussion here is informed by several sources; some modern (Chorin and Marsden (1993), Gonzalez and Stuart (2008), Majda and Bertozzi (2002) and Robinson et al. (2016)) and some less so (Sommerfeld (1950) and Stokes (1845)). We refer the reader to these texts and references therein for a more complete discourse.

The Navier–Stokes equations can be derived by modelling fluid in the interior of a domain as a continuum (i.e. we assume that the nuances of molecular interaction have a negligible effect on the macroscopic behaviour). For the purposes of the derivation, we will only consider smooth velocity fields.

As above, we denote the velocity of the fluid at time t and position x by $u(x,t)$. We will assume that the flow is volume preserving, i.e. that there is no net flow across the boundary of any compact subdomain $\Omega' \Subset \Omega$:

$$\int_{\partial\Omega'} u \cdot \mathrm{d}n = 0$$

where n denotes the outward unit normal. This corresponds to the pointwise constraint

$$\nabla \cdot u = 0.$$

It is natural to also require that mass is conserved, which can be formulated as

$$\partial_t \rho + \nabla \cdot (u\rho) = \partial_t \rho + (u \cdot \nabla)\rho = 0,$$

where $\rho(x,t)$ gives the distribution of density in the fluid. Equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega'} \rho = - \int_{\partial\Omega'} u\rho \cdot \mathrm{d}n$$

for any $\Omega' \Subset \Omega$.

We will only be considering the case of a homogeneous fluid, i.e. one with constant density. To save notation let us assume that $\rho \equiv 1$, then the mass-conservation constraint becomes a redundant copy of the incompressibility constraint.

Setting aside the extrinsic body forcing, we assume that two effects govern the motion of the fluid, namely pressure and viscous forces (friction). We will consider the total force acting on a region of fluid Ω' at one instant of time, which (by Newton's third law of motion) we assume to be the integral over the boundary $\partial\Omega'$ of the forces exerted there.

The force caused by the pressure is modelled as acting in the direction of the inward normal $(-n)$ to $\partial\Omega'$ at every point, with magnitude equal to the pressure at that point. This contributes the force

$$\int_{\partial\Omega'} -pn \,\mathrm{d}A = \int_{\Omega'} -\nabla p \,\mathrm{d}x.$$

To model the viscous forces, we assume that they are proportional to the rate of strain (defined below) across the boundary in the outward direction. The intuition here generalises Newton's model that friction between moving lamina is proportional to the derivative of the velocity, taken perpendicular to the lamina. Essentially, the rate of strain tensor is the component of the gradient of the velocity, that gives a first-order approximation of how the flow is"pulling apart", relative to any rigid motion.

In more detail, at a point $x \in \partial\Omega'$, we consider a linear approximation to the velocity:

$$u(x + \delta x, t) \approx u(x,t) + (\nabla u)\delta x = u(x,t) + \left[\partial_j u_i \delta x_j\right]_i,$$

for sufficiently small $\delta x \in \mathbb{R}^3$. We therefore approximate the evolution of the fluid relative to the motion at (x,t) by considering trajectories corresponding to a fixed velocity field v, given by:

$$v(y) = \nabla u(x,t) y.$$

Indeed let X and Y be Lagrangian trajectories corresponding to the flow u, such that $X(t) = x$ and $Y(t) = x + \delta x$, for some small $\delta x \in \mathbb{R}^3$, then

$X - Y$ evolves initially as

$$\frac{\mathrm{d}}{\mathrm{d}t}(Y(t) - X(t)) \approx (\nabla u)\delta x = v(Y(t) - X(t)). \tag{2.5}$$

Consider the anti-symmetric and symmetric components of ∇u. That is a rigid (or rotational) part

$$R(x,t) := \frac{1}{2}\left(\nabla u(x,t) - (\nabla u(x,t))^\top\right),$$

and an (elastic) strain part, (named in analogy with the theory of elastic solids)

$$E(x,t) := \frac{1}{2}\left(\nabla u(x,t) + (\nabla u(x,t))^\top\right).$$

We see that, as a first-order approximation, $X - Y$ evolves as

$$X(t+\tau) - Y(t+\tau) = \mathrm{e}^{\tau R}\mathrm{e}^{\tau E}(X(t) - Y(t))$$

for sufficiently small $\tau > 0$. Note that this decomposition of a velocity field locally into a translation, a rotation, and a strain is essentially an observation of Helmholtz (1858).

The anti-symmetric component R corresponds to a rigid (i.e. rotational) motion. Indeed it is straightforward to check that the system

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(s) = RZ(s)$$

describes an isometric evolution i.e.

$$|\mathrm{e}^{\tau R}z| = |z|$$

for all $\tau \in \mathbb{R}$ and all $z \in \mathbb{R}^d$. Hence evolution under $\mathrm{e}^{\tau R}$ does not contribute to the strain.

The remainder E we call the rate-of-strain tensor at (x,t). This can be further decomposed into a "rate-of-dilation" component

$$E_d = \frac{1}{3}\nabla \cdot uI,$$

which vanishes in the incompressible case, and a volume-preserving "rate-of-shear" component

$$E_s = E - E_d.$$

We can now make precise our modelling assumption that the viscous force is proportional to the rate of strain accross the boundary, namely we model the contribution of these forces on the region Ω' by

$$2\nu \int_{\partial\Omega'} E \cdot \mathrm{d}n,$$

where $\nu > 0$ (or $\nu = 0$ in the derivation of the Euler equations) acts as the relative weight of the viscous forces in the evolution. In other words 2ν is the constant of proportionality in the Newtonian model.

By the incompressibility constraint, the viscous force amounts to

$$\nu \int_{\partial\Omega'} [(\nabla u) + (\nabla u)^\top] \cdot \mathrm{d}n = \nu \int_{\Omega'} \Delta u + \nu \int_{\Omega'} \nabla(\nabla \cdot u) = \nu \int_{\Omega'} \Delta u.$$

Combining the above expressions for the two principle intrinsic forces in the model, and adding a fixed body-force $f : \Omega \to \mathbb{R}^3$, we arrive at the the following equation for the evolution of the momentum of the fluid in an arbitrary fixed region $\Omega' \Subset \Omega$:

$$\int_{\Omega'} \frac{\partial^2}{\partial s^2} X(x,s)\,\mathrm{d}x = \int_{\Omega'} \nu\Delta u - \nabla p + f\,\mathrm{d}x, \tag{2.6}$$

where $X(x,\cdot) : [t, t+\varepsilon) \to \Omega$, for some $\varepsilon > 0$, denotes the trajectory of the fluid that passes through the point x at time t (we will discuss trajectory maps in more detail later). The left-hand side of (2.6) is the net acceleration of the fluid in Ω'. For all $x \in \Omega$, $X(x,\cdot)$ satisfies the system

$$\frac{\partial}{\partial s} X(x,s) = u(X(x,s),s),\ \ X(x,t) = x,$$

for all $s \in [t, t+\varepsilon)$. Hence the acceleration at (x,t) can be expressed in terms of u as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} X(x,t) = \frac{\partial}{\partial t} u(X(x,t),t) + \partial_{x_i} u(X(x,t),t) \frac{\partial}{\partial t} X_i(x,t) = \partial_t u + (u \cdot \nabla)u.$$

The right hand side is often called the *material derivative* of u. Now (2.6) becomes

$$\int_{\Omega'} \partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p - f \,\mathrm{d}x = 0.$$

Since Ω' was arbitrary we arrive at the classical formulation of the Navier–Stokes (Euler) equations:

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f,$$

with the constraint

$$\nabla \cdot u = 0.$$

On a domain with boundary, the Navier–Stokes equations are usually studied with a *no-slip* Dirichlet boundary condition $u\big|_{\partial\Omega} = 0$. This is appropriate for analysis and turns out to be phenomenalogically reasonable. In the inviscid case it is most usual to take *no-flow* boundary conditions, i.e. the Neumann condition

$$u \cdot n = 0 \text{ on } \partial\Omega.$$

2.3 Remarks on the Euler equations

The Euler equations have been a subject of study for over 260 years and in recent years have received a lot of attention, both analytical and numerical, along with related systems such as the Navier–Stokes equations, models from Magnetohydrodynamics, and the surface quasi-geostrophic (SQG) equations. Some recent articles discussing the current state-of-affairs (some perhaps not considered surveys) include Bardos and Titi (2007), Bardos and Titi (2013), Constantin (2006), Constantin (2007), Gibbon (2008) and Yudovich (2006). The aim of the next few pages is to give a little context to some of the results in this thesis, but it should be clear that this is far from a complete picture of the field. Indeed, we shall only point out a few of the more significant and historical results in the study of the Euler equations. Much more thorough discussions can be found in the aforementioned articles and references therein.

The state of knowledge on the well-posedness problems for the Euler equations is very different in dimensions two and three. For example Yudovich (1963) proved that in two dimensions a unique weak solution exists for initial data $u_0 \in L^2$ with $\|\nabla \times u_0\|_{L^\infty} < \infty$.

More precisely, in the case of bounded domains in two dimensions with a no-flow boundary condition, he proved that a unique weak solution of the Euler equations exists for initial data u_0 such that $u = \nabla^\perp \phi$ with $\Delta\phi \in L^\infty$. The solution was constructed by finding the stream function ϕ, which is achieved by iteratively solving a first-order linear system with parameters depending on the previous iterate and applying Schauder's fixed point theorem.

Bardos (1972) proved several related results, in particular that if $u_0 \in V = \{v \in H^1(\Omega) : \nabla \cdot v = 0, \, v \cdot n\big|_{\partial\Omega} = 0\}$ for a bounded C^2 domain $\Omega \subset \mathbb{R}^2$, then (in the homogeneous case) for any $T > 0$ there exists a weak solution $u \in L^\infty(0,T;V) \cap C(0,T;L^2)$ and this solution is unique if $\nabla \times u_0 \in L^\infty(\Omega)$. See also Marchioro and Pulvirenti (1994) for another proof that an initial vorticity $\omega_0 \in L^\infty(\Omega)$ gives rise to a unique solution for the vorticity $\omega \in L^\infty(0,T;L^\infty)$, where T is an arbitrarily large time.

It is important to note that although these works give global well-posedness results for weak solutions, they do not apply to arbitrary weak solutions with minimal regularity i.e. $u \in L^\infty(0,T;L^2) \cap L^2(0,T;V)$. Indeed, the fact that L^2 weak solutions are not unique in two (or higher) dimensions follows from the celebrated works of Scheffer (1993), subsequently Shnirelman (1997), and more recently DeLellis and Szekelyhidi (2010).

In the first of these articles, Scheffer constructed a non-zero weak solution of the Euler equations with compact support on $\mathbb{R} \times \mathbb{R}^2$ (space and time). Shnirelman proved a similar result on the torus $\mathbb{T}^2$, but using a simpler construction. The work by DeLellis and Szekelyhidi makes use of convex integration to show that there exist bounded vector fields with compact support in $\mathbb{R}^d$ ($d \geq 2$) that, when taken as initial data, give rise to infinitely many weak solutions of the Euler equations that additionally satisfy certain local and strong energy inequalities. These examples are highly oscillatory and energy dissipation, so DeLellis and Szekelyhidi dubbed these "wild solutions".

It is convenient to mention here that in 2011 Wiedemann Wiedemann (2011) used the results of DeLellis and Szekelyhidi to prove that there exist (infinitely many) global weak solutions on $\mathbb{T}^d$ ($d \geq 2$) for any divergence-free initial data $u_0 \in L^2$ with decaying energy as $t \to \infty$. It is important to note that Wiedemann's solutions need not satisfy an energy inequality; in particular the energy increases discontinuously at $t = 0$.

In three dimensions no analogue of the results of Bardos or Yudovich has been found. It is not known whether there are sufficient conditions that can be added to the definition of a weak solution that imply uniqueness, while still allowing for the construction of global solutions.

Despite the absence a global existence result for solutions satisfying an energy inequality for arbitrary $u_0 \in L^2$ in three dimensions, several short time existence results are known for more regular data. For example, local classical solutions on $\mathbb{R}^3$ (for sufficiently regular initial data) were constructed as early as the 1920s by Günther (1926) and Lichtenstein (1925a,b, 1927).

In the setting of Sobolev spaces, Kato (1972) proved that the Euler equations are locally well-posed in $H^m(\mathbb{R}^3)$ for $m > 3$. This result was extended to bounded domains by Temam (1975). Local existence and uniqueness in the Hölder spaces $C^{k,\alpha}(\Omega)$ for $k \geq 1$ was proved by Bardos and Frisch (1976) in a wide class of unbounded domains.

Chapter 3 of Majda and Bertozzi (2002) contains a clean proof of local existence in $H^s(\mathbb{R}^3)$ for $s \in \mathbb{Z}$ with $s \geq [d/2]+2$, which applies in dimensions $d = 2, 3$. This can be extended to local well-posedness in $H^s(\mathbb{R}^3)$ for $s > d/2 + 1$ (or on a bounded C^{s+2} domain, for a suitable notion of a non-integer Sobolev space there), see for example, the result by Temam (1976), part of which can be stated as follows:

Theorem 2.1. *Fix $d \geq 2$ and $s > \frac{d}{2} + 1$, for any $u_0 \in H^s(\mathbb{R}^d)$ with $\nabla \cdot u = 0$ there exists $T > 0$ (independent of s) and a unique solution to the homogeneous Euler equations*

$$u \in L^\infty(0, T; H^s),\ p \in L^\infty(0, T; H^{s-1}).$$

If $d = 2$ we can take $T = \infty$.

A natural problem, given the current lack of global existence results in three dimensions (for the spaces in which solutions are unique), is finding suitable a priori estimates on solutions to prevent a blowup in finite time and allow extension to all times $t > 0$. The most renowned result in the arc of this problem is due to Beale, Kato, and Majda (1984). In that concise work it is proved that the L^∞ norm of the vorticity must blow up if the solution cannot be extended beyond some finite time. Formally, for a solution u, the vorticity $\omega = \nabla \times u$ satisfies the following *vorticity* *equations*:

$$\partial_t \omega + (u \cdot \nabla)\omega - \Delta\omega = (\omega \cdot \nabla)u,$$

in the three-dimensional case. In the case of spatial dimension two, ω is only scalar valued and the *vortex-stretching term* $(\omega \cdot \nabla)u$ does not appear in the analogous equations. The reason for this terminology is more clear in the Lagrangian setting (see the remarks below).

The Beale-Kato-Majda theorem can be stated as

Theorem 2.2. *Let $u \in C([0,T]; H^s(\mathbb{R}^3)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^3))$ be a strong solution of the homogeneous Euler equations for $s \geq 3$. Suppose that u cannot be extended to a solution on a larger time interval, then*

$$\int_0^T \|\omega(t)\|_{L^\infty} \, \mathrm{d}t = \infty$$

and, in particular

$$\limsup_{t \to T} \|\omega(t)\|_{L^\infty} = \infty.$$

Beale et al. (1984) claim that this result also applies on $\mathbb{T}^3$ and suitably smooth bounded domains, although they do not prove it in that paper.

In Chapter 3 we will consider a formulation of the Euler equations that mixes the Eulerian velocity u with certain Lagrangian quantities. In contrast, several authors have studied formulations that use Lagrangian variables and vorticity at once. For example, it is well known that for a C^1 solution of the Euler equations, the vorticity satisfies

$$\omega(X(a,t),t) = \nabla_a X \cdot \omega(a,0),$$

in dimension three, or

$$\omega(X(a,t),t) = \omega(a,0)$$

in dimension two. See, for example, Chapter 2 of Majda and Bertozzi (2002), or Chapter 2 of Marchioro and Pulvirenti (1994). In other words, the vorticity is transported by the flow u, but in three dimensions may be stretched and rotated by the gradient of the trajectory map. It is often pointed out in the literature that the presence, or otherwise, of vortex stretching is one of the key qualitative differences between the two and three dimensional cases.

2.4 Remarks on the Navier–Stokes equations

2.4.1 Global existence vs uniqueness in different function spaces

The Navier–Stokes equations were formulated by George Stokes over 170 years ago, and although they are younger than Euler equations, today their name is perhaps more well known within the wider mathematical community. If this is the case, it is likely due to the fact that the global existence and smoothness problem for (bounded energy) solutions of the three-dimensional Navier–Stokes equations, was included in the Clay Institute's list of seven prize problems for the new millennium in 2000.

The official statement of the problem was given by Fefferman (2006). It asks whether or not the Navier–Stokes equations admit a smooth global solution in $C^\infty(\Omega \times [0,\infty)) \cap L^\infty(0,\infty; L^2(\Omega))$ for any choice of initial data $u_0 \in C^\infty \cap L^2(\Omega)$, where $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$. If the answer is found to be negative, the challenge would be to find a counterexample, which may have non-zero forcing (so long as the forcing is sufficiently smooth).

Despite the popularity of the study of the Navier–Stokes equations, the millennium problem is very much open, at the time of writing.

Many books, surveys and expositions of standard theory have been written about the Navier–Stokes equations, for example Constantin (2008), Constantin and Foias (1988), Galdi (2000), Ladyzhenskaya (1969, 2003), Lemarié-Rieusset (2002), Robinson (2006a), Robinson et al. (2016), Sohr (2001), Temam (2001) and Yudovich (2006). As in the previous section we will now give a brief review of some of the most significant and historical results concerning well-posedness of the Navier–Stokes equations, without going into much detail or attempting to be comprehensive.

Perhaps the most renowned results on the three-dimensional Navier–Stokes equations are the whole-space global existence results for weak solutions and blowup estimates found by Leray (1934). It seems appropriate here to give a brief summary of that work. A far more detailed exposition by Ożański and Pooley (2016) is currently in preparation.

Leray begins by proving that sufficiently regular initial data in $\mathbb{R}^3$ gives rise to a unique classical solution of the Navier–Stokes equations for a short time, the construction is based on an iterative application of Oseen's fundamental solution (Oseen (1911)) of the following initial value problem for the Stokes equations:

$$\partial_t u - \Delta u + \nabla p = X,\ \nabla \cdot u = 0,\ u(x,0) = 0.$$

Next, Leray observes, using estimates from the construction, that if a solution u cannot be extended beyond a blowup time T, then the L^∞ norm must be unbounded at T, moreover he finds lower bound on the rate of blowup of the form

$$\|u(s)\|_{L^\infty} \geq c(T-s)^{-1/2}.$$

Similarly, he obtains a lower bound on the H^1 (semi)-norm:

$$\|\nabla u(s)\|_{L^2} \geq c(T-s)^{-1/4}.$$

The local existence and uniqueness is then generalised to initial data in H^1 (in the sense of classical solutions on $(0,T)$ with weak convergence to the initial data in L^2).

Leray constructs global weak solutions satisfying an energy inequality for divergence-free initial data in L^2. This is achieved by considering a sequence of linearised problems:

$$\partial_t u + [(\mathcal{J}_\varepsilon u) \cdot \nabla]u - \Delta u + \nabla p = 0$$

where $\mathcal{J}_\varepsilon$ denotes a mollification. As $\varepsilon \to 0$ it is shown that the solutions converge in L^2 to a weak solution of the Navier–Stokes equations.

The final chapter of Leray (1934) concerns the renowned *Théorèm de structure* (also called the *epochs of regularity property*), which can be summarised as follows. If u is a global Leray–Hopf weak solution of the Navier–Stokes equations then there exist finitely many open intervals (a_i, b_i) (b_i may be ∞) such that u is a classical solution on each interval and the union has full measure:

$$\lambda\left([0,\infty)\backslash \bigcup_i (a_i, b_i)\right) = 0.$$

Hopf (1951) later extended Leray's existence results to bounded domains $\Omega \subset \mathbb{R}^d$, for $d \geq 2$, using a more modern construction. A crucial feature of the weak solutions found by these authors is that they satisfy a certain energy inequality, in addition to the weak formulation of the equations. A weak solution u of the Navier–Stokes equations on $[0,T]$ is said to satisfy the energy inequality if

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq \|u(s)\|_{L^2}^2$$

for all $t \in [0,T]$. In recognition of these two contributions, weak solutions satisfying the energy inequality are called *Leray–Hopf weak solutions*.

Global existence for Leray–Hopf weak solutions has been dealt with in other unbounded domains; see for example Galdi and Maremonti (1986) for a treatment of exterior domains or, Heywood (1988) for the case of arbitrary domains in $\mathbb{R}^3$ with C^2 boundary.

Leray also essentially proved that his solutions in $\mathbb{R}^3$ satisfy the *strong energy inequality*, that is, for almost all $s > 0$

$$\|u(t)\|_{L^2}^2 + \int_s^t \|\nabla u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq \|u(s)\|_{L^2}^2$$

for all $t > s$.

Subsequent work by Ladyzhenskaya (see Chapter 6 of Ladyzhenskaya (1969)) showed that solutions satisfying the strong energy inequality could be constructed on bounded domains, extending Hopf's result. Further refinements regarding the strong energy inequality and epochs of regularity property for solutions in bounded and unbounded domains (other than $\mathbb{R}^3$) can be found in Heywood (1988) and references therein.

The definition of weak solutions and an example of these existence results in L^2 will be discussed in some detail below, with proofs following the modern literature. This is partly to highlight these results and partly to illustrate some techniques that will appear throughout this thesis.

An important consequence of the energy inequality is that a strong solution $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ is unique in the class of Leray–Hopf solutions. In other words, existence of a strong solution on [0,T) implies uniqueness for Leray–Hopf solutions on the same time interval. This type of result is called *weak-strong uniqueness*. For more information see, for example Constantin and Foias (1988) or Robinson et al. (2016).

At this point it is convenient to mention the fact that weak solutions are unique in two dimensions. To illustrate this, suppose u and v are two solutions on $\mathbb{R}^2$ and let $w = u - v$. We obtain good a priori estimates on $\|w(t)\|_{L^2}$ as follows. Integrating the equation satisfied by w against w yields (after typical manipulations - see the 3D energy estimates in the following section)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \leq |(w \cdot \nabla v, w)_{L^2}| \leq \|w\|_{L^4}^2 \|\nabla v\|_{L^2}.$$

By the two-dimensional Ladyzhenskaya inequality (Ladyzhenskaia (1959)), or equivalently a Gagliardo-Nirenberg inequality, the right-hand side is less than

$$\|w\|_{L^2}\|\nabla w\|_{L^2}\|\nabla v\|_{L^2} \leq \frac{1}{2}\|w\|_{L^2}^2\|\nabla v\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2.$$

Hence, if v has (formally) the regularity of a weak solution, in particular $v \in L^2(0,T;H^1)$, then it follows from a Gronwall lemma that

$$\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 \mathrm{e}^{\int_0^t \|\nabla v(s)\|_{L^2}^2 \ \mathrm{d}s},$$

so $u \equiv v$ if $u_0 = v_0$. For a rigorous discussion of uniqueness in two dimensions, see Constantin and Foias (1988), or Lions and Prodi (1959).

In three dimensions we do not have sufficiently good Sobolev embeddings to prove uniqueness for weak solutions of the Navier–Stokes equations (at least not in such a straightforward way). Indeed, in this context, the question “Are weak solutions are unique?” and its counterpart, “Can strong solutions blow up in finite time?”, have been popular open problems for a number of years.

Much effort has been spent studying the Navier–Stokes equations in intermediate spaces i.e. looking for smaller classes of functions such that a global solution starting in the class remains there for all positive time, and larger classes of functions in which solutions are unique. The existence and uniqueness problem can be thought of as finding a class with both of these properties.

Of particular importance are the spaces that are critical with respect to the natural scaling of the equations. It is easy to check that if (u,p) a (classical) solution on $\mathbb{R}^d \times [0,T)$ then

$$u_\lambda(x,t) = \lambda u(\lambda u, \lambda^2 t),\ p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

is also a solution on $\mathbb{R}^d \times [0, T/\lambda^2)$ for any $\lambda > 0$. A norm (or semi-norm)

$\|\cdot\|$ is said to be critical with respect to the scaling if

$$\|f_\lambda\| = \|f\|$$

for any function f on $\mathbb{R}^d$, where $f_\lambda(x) = \lambda f(\lambda x)$. Similarly a norm is *sub-critical* or *super-critical*, with respect to the Navier–Stokes scaling, if $\|f_\lambda\|$ is an increasing or decreasing function of λ, respectively.

In three dimensions the L^3 norm and the seminorm $\|\Lambda^{1/2}\cdot\|_{L^2}$ on $H^{1/2}(\mathbb{R}^d)$ are examples of critical norms (semi-norms) in the above sense.

In the sub-critical L^p spaces, $p > d$, Fabes, Jones, and Rivière (1972) proved local existence and uniqueness of solutions in $\mathbb{R}^d$ for $d > 2$. For the supercritical spaces $L^p(\mathbb{R}^d)$, $2 < p < d$ $(d = 3, 4)$, Calderón (1990) proved global existence but not uniqueness. In the same paper Calderón proved that a unique global solution exists, for sufficiently small initial data in $L^d(\mathbb{R}^d)$ and that a unique local solution exists for arbitrary initial data in $L^d(\mathbb{R}^d)$.

The situation is similar in $H^{1/2}(\mathbb{R}^d)$. For example, Fujita and Kato (1964) proved local well-posedness, and global well-posedness for small data in $\mathcal{D}(\Lambda^{1/2})$ (see also Marín-Rubio, Robinson, and Sadowski (2013)).

The best known global well-posedness result of this type is due to Koch and Tataru (2001). They proved that divergence-free initial data that is sufficiently small in the critical space BMO^{-1} on $\mathbb{R}^d$ gives rise to a unique global solution. Koch and Tataru give a definition of BMO^{-1} which they show is equivalent to the space of tempered distributions that can be realised as the divergence of a function in BMO. Their well-posedness result for small data is the best, in the sense that BMO^{-1} is the largest critical space in which such a result has been proved.

2.4.2 Partial regularity and trajectories avoiding singular sets

In this subsection we will highlight some well-known developments that give estimates on the size of the set of singularities. This will allow us to briefly describe an interesting result by Robinson and Sadowski (2009b) that almost every Lagrangian trajectory[1] exists and is C^1 for a certain type

[1] Of course for a weak solution, one must use an appropriate notion of a Lagrangian trajectory.

of weak solution (for which we have a global existence results). Since we will later discuss certain Lagrangian quantities for the Euler and Navier–Stokes equations, this is a compelling result to keep in mind.

It essentially follows from Leray's proof of the epochs of regularity property that the Hausdorff dimension of the set of singular times is at most 1/2 (see Galdi (2000) or Robinson et al. (2016)). Moreover Scheffer (1976) proved the stronger result that the 1/2-dimensional Hausdorff measure of the set is 0, in the case of the whole space. The latter result was extended to bounded domains by Foiaş and Temam (1979). Estimates on the dimension of the putative set of singular points in space and time have become known as *partial regularity results*. Scheffer attributes the first consideration of partial regularity for Navier–Stokes to Mandelbrot (1976).

A later work of Scheffer (1977) included several partial regularity results. In particular he showed that divergence-free initial data in L^2 gives rise to a weak solution[2] u, of the Navier–Stokes equations (with zero forcing or $f \cdot u \leq 0$) such that (after modification on a λ-null set) u is continuous outside a set $S \subset [0,\infty) \times \mathbb{R}^3$, with $\mathcal{H}^2(S) < \infty$, where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure.

A further refinement by Scheffer (1980) showed that, in the case of a bounded domain, this upper bound on the Hausdorff dimension of S could be improved to 5/3. These ideas were further developed in the celebrated work by Caffarelli, Kohn, and Nirenberg (1982). For example, a consequence of their main result is that given a divergence-free $u_0 \in L^2(\mathbb{R}^3)$ there exists a *suitable weak solution* of the (homogeneous) Navier–Stokes equations, and the corresponding singular set $S \subset \mathbb{R}^3 \times [0,\infty)$ satisfies $\mathcal{P}^1(S) = 0$. Here $\mathcal{P}$ is the analogue of the Hausdorff measure defined using parabolic cylinders, i.e. sets of the form

$$Q_r(x,t) := \{(y,s) : |y-x| < r,\ s \in (t-r^2, t)\}.$$

The full result also applies to certain bounded domains and certain forcing terms in the inhomogeneous case. A suitable weak solution is essentially a Leray–Hopf weak solution such that the corresponding $p \in L^{5/4}(\Omega \times [0,T))$

[2] Scheffer gives a slightly different definition of weak solution, in particular he does not take the test functions to be divergence free.

and for which the following local energy inequality holds:

$$2\int_{\Omega\times[0,T)}|\nabla u|^2\phi \leq \int_{\Omega\times[0,T)}|u|^2(\partial_t\phi+\Delta\phi)+(|u|^2+2p)(u\cdot\nabla)\phi \quad (2.7)$$

for any $\phi \in C_c^\infty([0,\infty)\times\mathbb{R}^3)$. Lin (1998) has written a concise review of these results with alternative proofs.

An interesting consequence of known partial regularity results has been proved by Robinson and Sadowski (2009a,b). In the first paper it is proved that if u is a suitable weak solution exists on a C^2 bounded domain $\Omega \subset \mathbb{R}^3$, and additionally $u \in L^{6/5}(0,T;L^\infty)$ then almost every Lagrangian trajectory avoids the singular set. Due to a result by Ladyzhenskaya and Seregin (1999) we may assume that u is Hölder continuous outside of the singular set, so another consequence is that almost every Lagrangian trajectory is C^1.

The second paper Robinson and Sadowski (2009b), from the same year, gives an improvement on this result that essentially allows the condition $u \in L^{6/5}(0,T;L^\infty)$ to be dropped. The refinement is based on the observation that the box-counting dimension of the singular set is at most $5/3$ (which is an adaptation of results mentioned above), and the fact that almost all Lagrangian trajectories in a (non-smooth) volume preserving flow on $\mathbb{R}^d$ avoid a set with box-counting dimension less than $d-1$. The latter fact is similar to results on flows with prescribed regularity "avoiding" sets with small dimension by Aizenman (1978), and Cipriano and Cruzeiro (2005).

2.5 Weak solutions of the Navier–Stokes equations

As an illustration of some of the standard techniques that we will use frequently in this thesis we will now give a brief exposition of the well known result that for given weakly divergence-free initial data $u_0 \in H(\mathbb{T}^d)$, the Navier–Stokes equations admit at least one weak solution

$$u \in L^\infty(0,\infty;L^2(\mathbb{T}^d)) \cap L^2(0,\infty;H^1(\mathbb{T}^d)).$$

We begin with a discussion of the definition of a weak solution.

To define weak solutions of the Navier–Stokes equations, one might naively propose to call u, p a weak solution on $\Omega_T := [0,T) \times \Omega$ for initial data $u_0 \in L^1_{\text{loc}}$ and forcing $f \in L^1_{\text{loc}}(\Omega_T)$ if

$$\int_{\Omega_T} u \cdot \partial_t \phi - \partial_j u \cdot \partial_j \phi - [(u \cdot \nabla)u + \nabla p - f] \cdot \phi = -\int_{\Omega} u_0 \cdot \phi(x,0)\,\mathrm{d}x \quad (2.8)$$

for any (vector valued) $\phi \in C_c^\infty(\Omega_T)$. In this form we could formally interpret the equations for u and p such that u, $\nabla u \in L^2_{\text{loc}}(\Omega_T)$, $\nabla p \in L^1_{\text{loc}}$, for example. In practice (and in accordance with standard approach of converting linear parabolic PDEs into bilinear operators on $H^1_0 \times H^1_0(\Omega)$, see Evans (2010)) we will choose the function spaces described below.

We observe that the incompressibility constraint makes the pressure term redundant in (2.8). Indeed if (u,p) is a classical solution of the Navier–Stokes equations then

$$0 = \partial_t(\nabla \cdot u) - \Delta(\nabla \cdot u) = -\nabla \cdot [(u \cdot \nabla)u + \nabla p - f],$$

and so, by the Helmholtz decomposition, the gradient part of ϕ is orthogonal to the terms in the equation (in L^2) and makes no contribution to (2.8). This motivates us to choose a smaller class of test functions, namely those in $C_c^\infty(\Omega_T)$ that are divergence free. Upon integrating the equations against such a test function ϕ, the gradient term vanishes by orthogonality, which leaves

$$\int_{\Omega_T} u \cdot \partial_t \phi - \partial_j u \cdot \partial_j \phi - [(u \cdot \nabla)u - f] \cdot \phi = -\int_{\Omega} u_0 \cdot \phi(x,0)\,\mathrm{d}x. \quad (2.9)$$

Thus we have removed the unknown p from the equation for u. The pressure can be recovered from a weak solution using the identity

$$-\Delta p = \nabla \cdot [(u \cdot \nabla)u - f].$$

In keeping with the popular notation for the function spaces used in the study of the Navier–stokes equations in L^2, we let

$$\mathcal{V}(\Omega) := \{\phi \in C_c^\infty(\Omega) : \nabla \cdot \phi \equiv 0\}$$

then let $H \subset L^2(\Omega)$ and $V \subset H^1_0(\Omega)$ be the closure of $\mathcal{V}$ in the L^2 norm and the H^1 norm, respectively. We may now give a first definition of a weak solution.

Definition 2.3. *For $T > 0$ and a smooth open domain $\Omega \subseteq \mathbb{R}^d$ ($d \geq 2$) or $\Omega = \mathbb{T}^d$, $u \in L^\infty(0,T;H(\Omega)) \cap L^2(0,T;V(\Omega))$ is a weak solution of the Navier–Stokes equations, corresponding to the initial data $u_0 \in H$ and forcing $f \in L^2(0,T;H)$, if it satisfies (2.9) for all divergence-free test functions $\phi \in C_c^\infty(\Omega_T)$.*

Of course a classical solution u with sufficiently rapid decay (if Ω is unbounded) that $u(t) \in H \cap V$ for all $t \in [0,T]$ is a weak solution in this sense. Conversely, a sufficiently smooth weak solution is a classical solution. Indeed if u is smooth enough that we can integrate by parts in (2.9), then for a test function of the form $\phi(x,t) = a(t)b(x)$, where $a \in C_c^\infty([0,T))$ and $b \in C_c^\infty(\Omega)$ with $\nabla \cdot b = 0$

$$\int_0^T \int_\Omega \{\partial_t u + (u \cdot \nabla u) - \Delta u - f\}\, a(t)b(x)\,\mathrm{d}x\,\mathrm{d}t = 0$$

hence (since a was arbitrary)

$$\int_\Omega \{\partial_t u + (u \cdot \nabla u) - \Delta u - f\}\, b(x)\,\mathrm{d}x = 0$$

for all $t \in [0,T)$. By an observation of Hopf (1951), this is enough to deduce that

$$\partial_t u + (u \cdot \nabla u) - \Delta u - f = -\nabla p \tag{2.10}$$

for some differentiable function $p : \Omega_T \to \mathbb{R}$. The idea of Hopf's proof is that for any smooth closed curve in the interior of Ω we can find a divergence-free vectorfield supported on a tubular ε-neighbourhood of the curve that approximates the tangent for any $\varepsilon > 0$. By the orthogonality hypothesis and taking the limit $\varepsilon \to 0$, we find that the left-hand side of (2.10) is a conservative field, hence a gradient.

It can be shown (see Galdi (2000)) that a weak solution u can be redefined on a set of times with measure zero, such that for all $s \in [0,T)$ and all $t \in (s,T)$ and any divergence-free $\phi \in C_c^\infty(\Omega_T)$

$$\begin{aligned}\int_s^t \int_\Omega u \cdot \partial_t \phi - \partial_j u \cdot \partial_j \phi - [(u \cdot \nabla)u - f] \cdot \phi \\ = \int_\Omega u(t) \cdot \phi(x,t)\,\mathrm{d}x - \int_\Omega u(s) \cdot \phi(x,s)\,\mathrm{d}x.\end{aligned} \tag{2.11}$$

and the $L^\infty(0,T;L^2)$ bound extends to every $t \in [0,T)$. In particular, choosing $\phi(x,\tau) = a(\tau)\psi(x)$ where $a(\tau) = 1$ for all $\tau \in [s,t]$ we see that

$$\int_s^t \int_\Omega \partial_j u \cdot \partial_j \psi + [(u\cdot\nabla)u - f]\cdot\psi = -\int_\Omega u(t)\cdot\psi(x,t)\,\mathrm{d}x + \int_\Omega u(s)\cdot\psi(x,s)\,\mathrm{d}x,$$

for any divergence-free $\psi \in C_c^\infty(\Omega)$.

After this modification, it is easy to check that $(u(t),v)_{L^2}$ is a continuous function of t for any $v \in L^2$, i.e. $u \in C_w([0,T);L^2)$. Indeed, due to the fact that $u \in H$ it is enough to consider $v \in H$ and (2.11) implies that $(u(t),\psi)_{L^2}$ is continuous for any divergence-free $\psi \in C_c^\infty(\Omega)$, weak continuity follows from the uniform L^2 bound and the density of such ψ functions in H.

2.6 Leray-Hopf weak solutions: global existence

Here we will give a proof that weakly divergence-free initial data in $L^2(\mathbb{T}^d)$, $d = 2, 3$ gives rise to a Leray–Hopf solution. This largely serves to illustrate a well known application of some of the techniques we will use later. Our method of proof is standard, see for example Robinson et al. (2016), Galdi (2000) or Constantin and Foias (1988).

Theorem 2.4. *Let $d = 2$ or 3, for any weakly divergence free $u_0 \in L^2(\mathbb{T}^d)$, and any $T > 0$ there exists a weak solution of the Navier–Stokes equations on $[0,T)$ that satisfies the strong energy inequality.*

We will prove this by constructing a sequence of Galerkin approximations, by truncating the equations to a finite number of Fourier modes. We will show that the approximations converge to a weak solution using the Aubin-Lions lemma. The proof can easily be adapted to bounded domains by using a different basis to construct the Galerkin approximations, for example the eigenfunctions of $-\Delta$ with Dirichlet boundary conditions on $\Omega \subset \mathbb{R}^d$ (see the approach of Constantin and Foias (1988) for more details).

Proof. For $n > 0$ we consider the following Fourier truncation of the Navier–Stokes equations

$$\partial_t u_n + \mathbb{P}P_n[(u_n \cdot \nabla)u_n] - \Delta u_n = 0, \tag{2.12}$$

$$u_n(0) = P_n u_0 = 0,\ u_n = P_n u_n,$$

where P_n is the Fourier truncation described in Section 1.2.6. Notice that $\mathbb{P}$ and P_n commute. This system can be written as the following set of ODEs on a finite number of Fourier modes

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(k) = -|k|^2\hat{u}_n(k) - \frac{\mathrm{i}}{(2\pi)^{d/2}} \sum_{\substack{|j|\le n,\\ |k-j|\le n}} \hat{u}_n(k-j)\cdot j\left[\hat{u}_n(j) - \frac{\hat{u}_n(j)\cdot k}{|k|^2}k\right]. \tag{2.13}$$

Denoting $\hat{u}_n \coloneqq \mathcal{F}u_n$, this is a system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n = L_n(\hat{u}_n) + B_n(\hat{u}_n, \hat{u}_n)$$

for a linear map L_n and a bilinear map B_n. By standard techniques from the theory of ODEs (see, for example Hartman (2002)) there exists a unique solution $\hat{u}_n$, at least on a small time interval $[0, T_n)$, where $T_n > 0$ depends only on L_n, B_n and $u_n(0)$.

We will next show that this solution can be extended onto $[0, T)$ for any $T > 0$, independent of n. To achieve this we will derive energy estimates from (2.12) to show that $\|u_n(t)\|_{L^2} \le \|P_n u_0\|_{L^2} \le \|u_0\|_{L^2}$ for all $t \in [0, T_n)$. This will be sufficient because it implies that the solution can be continued onto $[0, mT_n)$ for any $m \in \mathbb{N}$.

Indeed the solution $\hat{u}_n$ can be continued as long as each coefficient $\hat{u}_n(k)$ remains finite, which is certainly the case if $\|\hat{u}_n(t)\|_{\ell^2} < \infty$ (equivalently $\|u_n(t)\|_{L^2} < \infty$).

To obtain the necessary energy estimates we integrate (2.12) against $2u_n$, noting that

$$(\mathbb{P}P_n[(u_n\cdot\nabla)u_n], u_n)_{L^2} = ((u_n\cdot\nabla)u_n, \mathbb{P}P_n u_n)_{L^2} = ((u_n\cdot\nabla)u_n, u_n)_{L^2},$$

and the integration by parts:

$$-(\Delta u_n, u_n)_{L^2} = \|\nabla u_n\|_{L^2}^2.$$

Thus we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\|u_n(t)\|_{L^2}^2 + 2\|\nabla u_n(t)\|_{L^2}^2 = -2\int_{\mathbb{T}^d}[(u_n\cdot\nabla)u_n]\cdot u_n(x,t)\,\mathrm{d}x.$$

The right-hand side vanishes by the anti-symmetry relation[3]:

$$((u \cdot \nabla)v, w)_{L^2} = -((u \cdot \nabla)w, v)_{L^2}$$

for $u, v, w \in C^1(\mathbb{T}^3)$ and $\nabla \cdot u = 0$. Hence

$$\|u_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u_n\|_{L^2}^2 \le \|u_n(0)\|_{L^2}^2 \le \|u_0\|_{L^2}^2. \tag{2.14}$$

Moreover, for any $0 \le s \le t < T$

$$\|u_n(t)\|_{L^2}^2 + 2\int_s^t \|\nabla u_n\|_{L^2}^2 \le \|u_n(s)\|_{L^2}^2. \tag{2.15}$$

Fix any $T > 0$, we now show that u_n converges to a weak solution on $[0, T)$. From (2.14) it follows that u_n is bounded independent of n in $L^\infty(0, T; H)$ and $L^2(0, T; V)$. Moreover we can uniformly estimate the time derivatives; integrating (2.12) against an arbitrary $v \in V(\mathbb{T}^d)$ yields

$$|\langle \partial_t u_n, v\rangle| \le \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + |((u_n \cdot \nabla)u_n, v)|.$$

The nonlinear term can be estimated by

$$|((u_n \cdot \nabla)u_n, v)| = |((u_n \cdot \nabla)v, u_n)| \le c\|v\|_{H^1}\|u_n\|_{L^4}^2$$

If $d = 2$ or 3 we can use the interpolation

$$\|u_n\|_{L^4} \le \|u_n\|_{L^2}^{1/4}\|u_n\|_{L^6}^{3/4} \le C\|u_n\|_{L^2}^{1/4}\|u_n\|_{H^1}^{3/4},$$

to show that $\partial_t u_n$ is bounded in $L^{4/3}(0, T; V^*)$ independent of n.

Using the uniform bounds above we apply the Aubin–Lions lemma and so, after relabelling, we may assume that u_n converges to a limit u weakly in $L^2(0, T; V)$ and strongly in $L^2(0, T; H)$. Moreover $\partial_t u_m$ converges weakly to $\partial_t u$ in $L^{4/3}(0, T; V^*)$.

In two dimensions we can actually do a little better using the Ladyzhenskaya inequality:

$$\|u_n\|_{L^4} \le \|u_n\|^{1/2}\|\nabla u_n\|^{1/2},$$

[3]Similar relations hold for less regular functions, for example $v, w \in H^1$ and $u \in H$ if $|u||\nabla v||w|$ and $|u||\nabla w||v|$ belong to $L^1(\Omega)$.

so we find uniform bounds on $\partial_t u_n \in L^2(0,T;V^*)$.

To show that the limit u satisfies (2.9) we use the fact that for every n, u_n satisfies

$$\int_{\Omega_T} u_n \cdot \partial_t \phi - \partial_j u_n \cdot \partial_j \phi - [(u_n \cdot \nabla)u_n] \cdot \phi = 0$$

for any divergence-free $\phi \in C^\infty(\mathbb{T}^d \times [0,T))$ such that $\phi = P_m \phi$ for some $m \leq n$. Since u_n converges weakly in $L^2(0,T;V)$ it is clear that

$$\int_0^T \int_{\mathbb{T}^d} (u_n - u) \cdot \partial_t \phi - (\partial_j u_n - \partial_j u) \cdot \partial_j \phi \,\mathrm{d}x\,\mathrm{d}t \to 0$$

as $n \to \infty$. For the nonlinear term, by anti-symmetry it suffices to prove that

$$\int_0^T \int_{\mathbb{T}^d} (u_n \cdot \nabla)\phi \cdot u_n - (u \cdot \nabla)\phi \cdot u \,\mathrm{d}x \to 0,$$

which is straightforward because the absolute value of this integral is less than

$$\|u_n - u\|_{L^2(0,T;H)} \|\nabla \phi\|_{L^\infty(0,T;L^\infty)} \left[\|u_n\|_{L^2(0,T;H)} + \|u\|_{L^2(0,T;H)}\right] \to 0.$$

We now see that the limit u satisfies (2.9) for any test-function ϕ with only finitely many non-zero Fourier modes. We can extend this to any divergence-free $\phi \in C^\infty(\mathbb{T}^d)$. Indeed, $\phi - P_m\phi$ is bounded in $C^1([0,\tau];H^s)$ if $0 < \tau < T$ for any $s \geq 0$. Hence

$$\|\phi - P_m \phi\|_{L^\infty(0,T;L^\infty)} \to 0$$

as $m \to \infty$, so the nonlinear terms with $P_m\phi$ as test functions converge to the nonlinear term for ϕ:

$$\int_0^T \int_{\mathbb{T}^d} (u \cdot \nabla)u \cdot (\phi - P_m\phi)$$
$$\leq \|u\|_{L^2(0,T;H)} \|u\|_{L^2(0,T;V)} \|\phi - P_m\phi\|_{L^\infty(0,T;L^\infty)} \to 0.$$

Convergence for the other terms is more straightforward.

We have now proved that u is a weak solution, hence $u \in C_w([0,T);L^2)$. It remains to show that the strong energy inequality holds. Passing to a subsequence if necessary we may assume that $\|u_n(t)\|_{L^2} \to \|u(t)\|_{L^2}$ for almost every $t \in [0,T)$. Taking the $\liminf$ in (2.15) as $n \to \infty$ yields

$$\|u(t)\|_{L^2}^2 + 2\int_s^t \|\nabla u\|_{L^2}^2 \leq \|u(s)\|_{L^2}^2 \tag{2.16}$$

for $0 \leq s \leq t < T$ such that $u_n(t)$, $u_n(s)$ converge to $u(t)$, $u(s)$ in L^2, respectively. Moreover, if (2.16) holds for some s and any $t > s$ then it also holds for all $t \in [s,T)$. Indeed, fixing a sequence of times $t_k \to t$ such that (2.16) holds at each t_k and taking the $\liminf$ as $k \to \infty$, we see that this inequality holds at time t. $\square$

This concludes our discussion of the mathematical background for the Euler and Navier–Stokes equations. In the remaining chapters we discuss the main content of this thesis.

Chapter 3

Eulerian-Lagrangian formulations of the Euler and Navier–Stokes equations

3.1 Introduction

In this chapter we study the so-called "Eulerian-Lagrangian" formulation of the Euler equations, as discussed by Constantin (2000) who has proved a local well-posedness result for the system in $C^{1,\mu}$. Instead, we will work in the corresponding Sobolev spaces H^s for $s > 1 + d/2$, for dimensions $d \geq 2$ and prove an analogous result using different estimates. The bulk of this chapter is based on a paper by Pooley and Robinson (2016b). In the last section we will make some remarks about possible Eulerian-Lagrangian formulations of the Navier–Stokes equations that will eventually lead us to the magnetization variables formulation, discussed in Chapter 6.

To keep the analysis as simple as possible we will work on $\mathbb{T}^d$ in the absence of external forcing. Recall that the incompressible Euler equations comprise the system

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \tag{3.1}$$

with

$$\nabla \cdot u \equiv 0.$$

Constantin has studied a form for the Euler equations that involves both the classical velocity field and the so called back-to-labels map A which is defined to be the inverse of the trajectory map X at each time t. More precisely, for an evolving vector field u defined on $\mathbb{T}^d \times [0,T]$, the trajectory map solves

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(y,t) = u(X(y,t),t) \\ \\ X(y,0) = y \end{cases} \tag{3.2}$$

for each $y \in \mathbb{T}^d$. If u is divergence free and sufficiently regular then X is well defined and $X(\cdot,t)$ is bijective for each t. In this case we can define the back-to-labels map A by setting

$$A(\cdot,t) \coloneqq X^{-1}(\cdot,t), \tag{3.3}$$

where we consider X as a map $X(\cdot,t) : \mathbb{T}^d \to \mathbb{T}^d$ for each $t \in [0,T]$. For the Eulerian-Lagrangian formulation, Constantin (2000) proved local existence and uniqueness results in certain Hölder spaces on $\mathbb{R}^3$ for solutions that are periodic, or satisfy suitable decay conditions.

As Yudovich (2006) noted, a similar combination of Eulerian and Lagrangian approaches was used to investigate the Euler equations in Hölder spaces, by Günther and Lichtenstein independently, as early as the 1920s (Lichtenstein (1927), Günther (1926)).

First we will review the Eulerian-Lagrangian formulation and discuss how it is formally equivalent to the usual Euler equations.

The main topic of this chapter is a proof of a local existence and uniqueness result for the Eulerian-Lagrangian formulation in $C([0,T];H^s(\mathbb{T}^d))$ with $s > \frac{d}{2}+1$ in dimension $d \geq 2$. The proof is self contained, in the sense that it neither appeals to results about the classical Euler equations, nor to the problem in Hölder spaces.

Following the first proof, we briefly discuss an alternative iteration scheme that follows more closely Constantin's argument in $C^{1,\mu}$. This requires a result estimating the composition of an H^s function with a volume-preserving H^s change of coordinates.

We end the chapter with some remarks on possible analogous formulations in the diffusive case.

3.2 The Eulerian-Lagrangian form of the equations

The Eulerian-Lagrangian form of the Euler equations comprises the following system:

$$\partial_t A + (u \cdot \nabla) A = 0, \tag{3.4}$$

$$u = \mathbb{P}((\nabla A)^\top v), \tag{3.5}$$

$$\partial_t v + (u \cdot \nabla) v = 0. \tag{3.6}$$

Given an initial divergence-free velocity u_0 for the classical equations, we choose initial conditions for the above system as follows:

$$A(x, 0) = x, \tag{3.7}$$

$$u(x, 0) = v(x, 0) = u_0(x). \tag{3.8}$$

The vector field v is called the *virtual velocity* and represents the initial velocity transported by the flow.

It will often be convenient to treat A as a perturbation of the identity map on $\mathbb{T}^d$. In this case we use the notation $\eta(x,t) := A(x,t) - x$ and replace (3.4) and (3.7) with the equations

$$\partial_t \eta + (u \cdot \nabla)\eta + u = 0, \quad \eta(x, 0) = 0 \tag{3.9}$$

respectively. We do this because the identity map (hence A) does not have sufficient Sobolev regularity when considered as a function on the torus with values in $\mathbb{R}^d$ (i.e. without accounting for the topology of the target torus).

As shorthand for the function space $C([0,T]; H^s(\mathbb{T}^d))$ in this chapter, we define $\Sigma_s(T)$ (usually denoted Σ_s) for $T \geq 0$ and $s \geq 0$ by

$$\Sigma_s(T) := C([0,T]; H^s(\mathbb{T}^d)).$$

We consider the natural norm on Σ_s:

$$\|u\|_{\Sigma_s} = \sup_{t \in [0,T]} \|u(t)\|_{H^s}.$$

We can now state the main result of this chapter is the following local well-posedness theorem, which we will prove in Section 3.3.

Theorem 3.1. *If $d \geq 2$, $s > \frac{d}{2}+1$ and $u_0 \in H^s$ is divergence free then there exists $T > 0$, such that the system (3.4–3.6) with initial conditions (3.7) and (3.8) has a unique solution A, u, v and $\eta, u, v \in \Sigma_s(T) \cap C^1([0,T]; H^{s-1})$ where $\eta(x,t) = A(x,t) - x$. Moreover $A \in C^1([0,T] \times \mathbb{T}^d)$ as a map into the torus.*

Before beginning the proof we will prove two propositions that make concrete the derivation and equivalence of the Eulerian-Lagrangian formulation with the classical Euler equations for functions with precisely defined regularity. These essentially constitute a more careful version of the derivation by Constantin (2000).

Proposition 3.2. *Let $d \geq 2$, and fix $u \in C^1((0,T) \times \mathbb{T}^d)$, with initial data $u(0) \in C^1(\mathbb{T}^d)$. If u is divergence-free and satisfies (3.1) for some p, with spatially periodic boundary conditions then $A \in C^1((0,T) \times \mathbb{T}^d; \mathbb{T}^d)$ and u satisfies (3.5) with $v(x,t) = u_0(A(x,t))$.*

Proof. From the regularity assumptions on u and periodicity of the domain we deduce that the trajectories $X(y,\cdot) \in C^2(0,T)$ and $\nabla X(y,\cdot) \in C^1(0,T)$ for all $y \in \mathbb{T}^d$, we also have $X, \frac{\partial X}{\partial t} \in C^1((0,T) \times \mathbb{T}^d)$. It follows from the divergence-free condition that $\det \nabla X \equiv 1$, so X is volume preserving and locally injective, hence bijective, given that $\mathbb{T}^d$ has finite volume. By the inverse function theorem we see that A exists and is an element of $C^1((0,T) \times \mathbb{T}^d)$. We now have enough regularity to make the following calculations rigorous.

From (3.1) and (3.2) we obtain

$$\frac{\partial^2 X}{\partial t^2}(y,t) = -\nabla p(X(y,t),t),$$

which is of course just a Lagrangian interpretation of the Euler equations. Setting $\tilde{p}(y,t) = p(X(y,t),t)$ this becomes

$$\frac{\partial^2 X}{\partial t^2} = -((\nabla X)^\top)^{-1} \nabla \tilde{p}(y,t).$$

Multiplying through by $(\nabla X)^\top$ and changing the order of differentiation yields

$$\frac{\partial}{\partial t}\left[\frac{\partial X_j}{\partial t}\frac{\partial X_j}{\partial y_i}\right] = \frac{\partial}{\partial y_i}\left[-\tilde{p} + \frac{1}{2}\left|\frac{\partial X}{\partial t}\right|^2\right] \tag{3.10}$$

for $i = 1, \ldots, d$, where there is an implicit sum over $j = 1, \ldots, d$ and X_j, y_i denote the components in $\mathbb{R}^d$ of X, y respectively. Integrating (3.10) in time, multiplying the corresponding vector equation by $(\nabla A)^\top$ and evaluating at $A(x,t)$ gives

$$u(x,t) = \frac{\partial X}{\partial t}(A(x,t),t) = (\nabla A)^\top u_0(A(x,t)) - \nabla n \tag{3.11}$$

where

$$n(x,t) = \int_0^t \tilde{p}(A(x,t),s) - \frac{1}{2}\left|\frac{\partial X}{\partial t}(A(x,t),s)\right|^2 \mathrm{d}s.$$

As gradients lie in the kernel of the Leray projector, applying $\mathbb{P}$ to (3.11) shows that u satisfies (3.5) as required. Note that $v(x,t) = u_0(A(x,t))$ satisfies (3.6), hence solutions to the Euler equations indeed solve the Eulerian-Lagrangian form. □

The converse is a little more technical.

Proposition 3.3. *Let $s > \frac{d}{2} + 1$. If*

$$u,\ v,\ \eta \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$$

satisfy (3.5), (3.6), (3.8) and (3.9), then there exists $p \in C([0,T]; H^s)$ such that (u,p) solves (3.1).

Proof. Since $H^{s-1}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ is an algebra, we have that if $f, g \in H^{s-1}$ (scalar valued) then

$$\partial_i(fg) = (\partial_i f)g + f(\partial_i g)$$

as an equality of L^2 functions, for $i = 1, 2, \ldots, d$. Therefore, denoting the material derivative by $\mathrm{D}_t \coloneqq \partial_t + (u \cdot \nabla)$, for $f, g \in C([0,T]; H^{s-1}) \cap C^1([0,T]; H^{s-2})$ we have

$$\mathrm{D}_t(fg) = (\mathrm{D}_t f)g + f(\mathrm{D}_t g). \tag{3.12}$$

Moreover, if $f \in H^s$,

$$(u \cdot \nabla)\nabla f = \nabla((u \cdot \nabla)f) - (\nabla u)^\top \nabla f.$$

Hence the classical commutation relation

$$\mathrm{D}_t \nabla f = \nabla \mathrm{D}_t f - (\nabla u)^\top \nabla f \tag{3.13}$$

holds as an equality in L^2, when $f \in C([0,T];H^s) \cap C^1([0,T];H^{s-1})$.

Since u satisfies (3.5), we may write

$$u(x,t) = v + (\nabla \eta)^\top v - \nabla q \tag{3.14}$$

for some $q \in C([0,T];H^s) \cap C^1([0,T];H^{s-1})$. Then by (3.12) and (3.13) the following calculations are justified (as equalities of L^2 functions for each $t \in [0,T]$):

$$\begin{aligned} \mathrm{D}_t u &= \mathrm{D}_t v + (\mathrm{D}_t \nabla \eta)^\top v + (\nabla \eta)^\top \mathrm{D}_t v - \mathrm{D}_t \nabla q \\ &= (\nabla \mathrm{D}_t \eta)^\top v - (\nabla u)^\top (\nabla \eta)^\top v - \nabla \mathrm{D}_t q + (\nabla u)^\top \nabla q \\ &= -(\nabla u)^\top [v + (\nabla \eta)^\top v - \nabla q] - \nabla \mathrm{D}_t q \\ &= -(\nabla u)^\top u - \nabla \mathrm{D}_t q \\ &= -\nabla p \end{aligned} \tag{3.15}$$

where $p = \frac{1}{2}|u|^2 + \mathrm{D}_t q$. A priori, we only have $D_t q \in C([0,T];H^{s-1})$, however since $D_t u \in C([0,T];H^{s-1})$, we see that, in fact, $p \in C([0,T];H^s)$.
□

3.3 Proof of Theorem 3.1

We will prove the local well-posedness result, Theorem 3.1 by constructing a contracting iteration scheme using the equations (3.5), (3.6) and (3.9). More precisely, given $u \in \Sigma_s(T)$ we find $v, \eta \in \Sigma_s \cap C^1([0,T] \times \mathbb{T}^d)$, solutions of

$$\partial_t \eta + (u \cdot \nabla)\eta = -u, \ \eta(0,x) = 0$$

and

$$\partial_t v + (u \cdot \nabla) v = 0, \ v(0,x) = u_0(x).$$

We then construct the next iterate of u, using

$$u' = \mathbb{P}[(\nabla A)^\top v]$$

and show that $u \mapsto u'$ is a contraction on a certain subset of Σ_s.

In the case of Hölder spaces, Constantin constructed an iteration scheme that was instead a contraction with respect to A. This involves controlling differences between candidate virtual velocities (v_1 and v_2, say) in terms of the difference between the respective back-to-labels maps (A_1 and A_2). This can be achieved, using the fact that $v_i = u_0(A_i)$ is a solution to (3.6). In the Hölder setting this is a natural way to proceed, however, relying on this *a posteriori* knowledge about the solution introduces an extra technicality when we work in Sobolev spaces. For this reason we will proceed as described above, relying only on a priori estimates. Following the proof, we shall see how the argument differs if the contraction is with respect to A, in particular we obtain an alternative proof under the additional assumption that $s \in \mathbb{Z}$.

We begin the proof of Theorem 3.1 by stating two inequalities concerning the advection term $(u \cdot \nabla)v$, using the notation $B(u,v) := (u \cdot \nabla)v$. Both of these results can be proved following the steps in Constantin and Foias (1988) or Robinson, Sadowski, and Silva (2012) (the only difference being that B here does not include a Leray projection).

Lemma 3.4. *For $s > \frac{d}{2}$ there exists $C_1 > 0$ such that if $u \in H^s$ and $v \in H^{s+1}$ then $B(u,v) \in H^s$ and*

$$\|B(u,v)\|_{H^s} \leq C_1 \|u\|_{H^s} \|v\|_{H^{s+1}}. \tag{3.16}$$

This is really just the fact that H^s is a Banach algebra. For the second lemma the assumption that u is divergence-free allows us to "save a derivative" by means of the identities

$$(B(u, (-\Delta)^{r/2} v), (-\Delta)^{r/2} v)_{L^2} = 0$$

for $r \in [0, s]$.

Lemma 3.5. *If $s > \frac{d}{2} + 1$ there exists $C_2 > 0$ such that for $u \in H^s$, $v \in H^{s+1}$ with u divergence free we have*

$$|(B(u,v), v)_{H^s}| \leq C_2 \|u\|_{H^s} \|v\|_{H^s}^2. \tag{3.17}$$

We use the following shorthand for closed balls in Σ_s:

$$B_M = \overline{B_{\|\cdot\|_{\Sigma_s}}(0, M)},$$

i.e. B_M is the closed unit ball centred at the origin of radius $M > 0$ with respect to the norm $\|\cdot\|_{\Sigma_s}$. Where ambiguity could arise we write $B_M(T)$ for the closed ball in $\Sigma_s(T)$.

Lemma 3.6. *If $s > \frac{d}{2} + 1$ and $\eta, v \in \Sigma_s(T)$ then $\mathbb{P}[(\nabla\eta)^\top v] \in \Sigma_s$ and there exists a constant $C_3 > 0$ (independent of η, v, t and T) such that for fixed t,*

$$\|\mathbb{P}[(\nabla\eta)^\top v]\|_{H^r} \leq C_3 \|\eta\|_{H^s} \|v\|_{H^r}, \tag{3.18}$$

where $r = s$ or $r = s - 1$. Furthermore, there exists $C_3' > 0$ such that for any $M > 0$ and $T > 0$, the following bounds hold uniformly with respect to $t \in [0, T]$ for any $\eta_1, \eta_2, v_1, v_2 \in B_M(T)$:

$$\|\mathbb{P}[(\nabla\eta_1)^\top v_1 - (\nabla\eta_2)^\top v_2]\|_X \leq C_3' M(\|\eta_1 - \eta_2\|_X + \|v_1 - v_2\|_X). \tag{3.19}$$

where X is $L^2(\mathbb{T}^d)$ or $H^{s-1}(\mathbb{T}^d)$.

Proof. For continuity into H^{s-1} we use the fact that H^{s-1} is a Banach algebra. More precisely, we see that

$$\begin{aligned}\|\mathbb{P}[(\nabla\eta_1)^\top v_1 - (\nabla\eta_2)^\top v_2]\|_{H^{s-1}} \leq\ & C\|\eta_1 - \eta_2\|_{H^s}\|v_1 + v_2\|_{H^{s-1}} \\ & + C\|\nabla\eta_1 + \nabla\eta_2\|_{H^{s-1}}\|v_1 - v_2\|_{H^{s-1}},\end{aligned} \tag{3.20}$$

where $C > 0$ is independent of the η_i and v_i. The key step in the proof of (3.18) when $r = s$ is that if $\eta, v \in C^2$ then for some $q \in H^s$,

$$\begin{aligned}\partial_{x_i}\mathbb{P}[(\nabla\eta)^\top v] &= \partial_{x_i}(\partial_{x_j}\eta_k v_k) - \partial_{x_i}\partial_{x_j} q \\ &= \partial_{x_j}(\partial_{x_i}\eta_k v_k) - \partial_{x_i}\eta_k \partial_{x_j} v_k + \partial_{x_j}\eta_k \partial_{x_i} v_k - \partial_{x_i}\partial_{x_j} q\end{aligned}$$

where sums are taken implicitly over k. The left-hand side is already divergence-free so projecting again removes the gradient terms and yields

$$\partial_{x_i}\mathbb{P}[(\nabla\eta)^\top v] = \mathbb{P}[(\nabla\eta)^\top \partial_{x_i} v - (\nabla v)^\top \partial_{x_i}\eta]. \tag{3.21}$$

By continuity, this still holds if we only have $\eta, v \in H^s$. A calculation similar to (3.20) applied to (3.21) yields continuity with respect to the H^s

norm as claimed.

The inequalities (3.18) for $r = s-1$ and $r = s$ are obtained by taking the H^{s-1} norms of $\mathbb{P}[(\nabla\eta)^\top v]$ and (3.21) respectively.

To prove (3.19), we again use the fact that $\mathbb{P}$ removes gradients. Indeed for[1] f, $g \in H^s$, we have

$$\mathbb{P}((\nabla f)^\top g) = \mathbb{P}(\nabla(f \cdot g) - (\nabla g)^\top f) = -\mathbb{P}((\nabla g)^\top f). \tag{3.22}$$

Setting $f = \eta_1 - \eta_2$, $g = v_1 + v_2$, we see that the calculations in (3.20) can be modified to give the required result. Note that for the L^2 bound we use the fact that (3.20) holds if we replace H^s with L^∞ and H^{s-1} with L^2. $\square$

The next lemma gives uniform bounds on the H^s norms of solutions to the transport equations (3.4) and (3.6). We will consider the following system:

$$\begin{cases} \partial_t f + (u \cdot \nabla) f = g \\ f(0) = f_0 \end{cases} \tag{3.23}$$

for functions $f, g : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ and a given divergence-free function u.

Lemma 3.7. *Let $s > \frac{d}{2} + 1$ and fix $f_0 \in H^s$, $g \in \Sigma_s$. If $u \in \Sigma_s$ is non-zero and divergence free then there exists a unique solution f to (3.23). Furthermore, the solution $f \in \Sigma_s \cap C^1([0,T]; H^{s-1}) \cap C^1([0,T] \times \mathbb{T}^d)$ and there exists $C_4 > 0$ (from Lemma 3.5) such that if $r, t \in [0,T]$ we have:*

$$\|f(t)\|_{H^s} \le \left(\|f(r)\|_{H^s} + \frac{\|g\|_{\Sigma_s}}{C_4 \|u\|_{\Sigma_s}} \right) \exp(C_4 |t-r| \|u\|_{\Sigma_s}) - \frac{\|g\|_{\Sigma_s}}{C_4 \|u\|_{\Sigma_s}}. \tag{3.24}$$

Proof. Using the method of characteristics we can construct a solution $f \in C^1([0,T] \times \mathbb{T}^d)$. The formal argument that follows motivates our consideration of the regularity of f. Taking the H^s product of (3.23) with f yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{H^s}^2 = -(B(u,f), f)_{H^s} + (f,g)_{H^s}.$$

By Lemma 3.5, there exists $C > 0$ such that for all $t \in [0,T]$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{H^s}^2 \le C\|u(t)\|_{H^s}\|f(t)\|_{H^s}^2 + \|g(t)\|_{H^s}\|f(t)\|_{H^s}. \tag{3.25}$$

[1]In this chapter the notation f will be used for several purposes and does not imply any connection with the body forcing of an inhomogeneous system.

Now (3.24) follows from Gronwall's inequality. In the case $r > t$, this argument is applied to the time-reversed equation, that is, using the fact that for fixed r, $-f(r-t)$ is transported by $-u(r-t)$ with forcing $g(r-t)$.

To properly justify this we can proceed by a Galerkin method. For each $N \in \mathbb{N}$ we find a solution to the system

$$\begin{cases} \partial_t f_N + P_N B(u_N, f_N) = g_N \\ f_N(r) = P_N f(r), \end{cases} \tag{3.26}$$

on $[r, T]$, where P_N denotes truncation up to Fourier modes of order N (in space), $u_N \coloneqq P_N u$ and $g_N \coloneqq P_N g$. The estimate (3.24) applies to f_N so by a standard argument using the Aubin-Lions lemma we obtain a weak solution $h \in L^\infty(r, T; H^s)$ such that $\partial_t h \in L^\infty(r, T; H^{s-1})$, hence $h \in C([0, T]; H^{s-1})$. Using the divergence free property we obtain uniqueness of solutions $h \in L^2(r, T; H^1)$ with time derivative $\partial_t h \in L^2(r, T; L^2)$. Indeed, if h and $\tilde{h}$ are two such solutions it follows from (3.23) that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|h - \tilde{h}\|_{L^2}^2 = 0.$$

Therefore $f = h$, i.e. this weak solution agrees with our C^1 classical solution on $[r, T]$.

We now prove (3.24) in the case $r \leq t$. As $f_N \to f$ in $L^2(r, T; H^{s-1})$, we may assume that $f_N(t_k) \to f(t_k)$ in H^{s-1} as $N \to \infty$, for each t_k in a dense countable subset $\{t_k\}_{k=1}^\infty \subset [r, T]$. The formal argument above is valid on the truncated system, thus

$$\|f_N(t_k)\|_{H^s} \leq \left(\|P_N f(r)\|_{H^s} + \frac{\|g\|_{\Sigma_s}}{C\|u_N\|_{\Sigma_s}}\right)\exp(C|t_k - r|\|u\|_{\Sigma_s}) - \frac{\|g_N\|_{\Sigma_s}}{C\|u\|_{\Sigma_s}}. \tag{3.27}$$

Hence, passing to a subsequence of f_N for each k with a diagonalisation argument, we may assume that for all k, $f_N(t_k)$ converges weakly in H^s as $N \to \infty$. Moreover, by the choice of the points t_k and uniqueness of weak limits, we must have $f_N(t_k) \rightharpoonup f(t_k)$ in H^s. Taking the lim inf of (3.27) with respect to $N \to \infty$ yields

$$\|f(t_k)\|_{H^s} \leq \left(\|f(r)\|_{H^s} + \frac{\|g\|_{\Sigma_s}}{C\|u\|_{\Sigma_s}}\right)\exp(C|t_k - r|\|u\|_{\Sigma_s}) - \frac{\|g\|_{\Sigma_s}}{C\|u\|_{\Sigma_s}}. \tag{3.28}$$

To prove (3.24) and the weak continuity of f into H^s we will use the fact that a weakly convergent sequence in H^{s-1} that is also bounded in H^s must converge weakly in H^s to the same limit by the Banach–Alaoglu theorem. Indeed if $x_k \rightharpoonup x$ in H^{s-1} is bounded in H^s then any subsequence admits a further subsequence converging weakly in H^s to x by the uniqueness of weak limits.

From this, (3.24) follows by the density of $\{t_k\}$ and the continuity of f into H^{s-1}. Indeed, in the case $t \geq r$, for any subsequence $(t_{k_\ell})_{\ell=1}^{\infty} \subset (t_k)_{k=1}^{\infty}$ such that $t_{k_\ell} \to t$ we have $f(t_{k_\ell}) \rightharpoonup f(t)$ in H^s. Applying (3.28) at t_{k_ℓ} and taking the $\liminf$ as $\ell \to \infty$ yeilds (3.24) at time t. For $t < r$ the required bounds are obtained in the same way from the time-reversed version of (3.26).

We have shown that $\|f(t)\|_{H^s}$ is bounded uniformly, not merely almost everywhere. Therefore for any fixed $\tau \in [0,T]$ and any sequence $\{\tau_k\} \subset [0,T]$ such that $\tau_k \to \tau$ we deduce, by the continuity into H^{s-1}, that $f(\tau_k) \rightharpoonup f(\tau)$ in H^s. This says that f is weakly continuous into H^s.

To see that $f \in \Sigma_s$ it is therefore enough to show that $\|f(t)\|_{H^s}$ is continuous. This is the case since for all $r, t \in [0,T]$, (3.24) gives bounds of the form

$$(\|f(r)\|_{H^s} + \alpha)\mathrm{e}^{-\beta|t-r|} - \alpha \leq \|f(t)\|_{H^s} \leq (\|f(r)\|_{H^s} + \alpha)\mathrm{e}^{\beta|t-r|} - \alpha$$

for time independent constants $\alpha, \beta > 0$, where the first inequality comes from (3.24) with r and t interchanged.

The fact that $f \in C^1([0,T]; H^{s-1})$ follows from the fact that $\partial_t f \in \Sigma_{s-1}$ which can be seen from the regularity of the other terms in (3.23). $\square$

Lemma 3.8. *For $s > d/2+1$ fix u_1, $u_2 \in \Sigma_s$ and $f_0 \in H^s$. Let $g_1 = g_2 = 0$ or $g_i = -u_i$ for $i = 1, 2$. If f_1, f_2 are the solutions of (3.23) corresponding to u_1, u_2, g_1, g_2 respectively, then in the case that $g_1 = g_2 = 0$, there exists $C_5 > 0$ depending only on s such that*

$$\|f_1(t) - f_2(t)\|_{L^2} \leq C_5 \|f_1 + f_2\|_{\Sigma_s} \|u_1 - u_2\|_{\Sigma_0} t \tag{3.29}$$

for all $t \in [0,T]$. In the case that $g_i = -u_i$ for $i = 1, 2$ we instead have

$$\|f_1(t) - f_2(t)\|_{L^2} \leq (C_5 \|f_1 + f_2\|_{\Sigma_s} + 1) \|u_1 - u_2\|_{\Sigma_0} t \tag{3.30}$$

Proof. Using the anti-symmetry of $(B(u_1 - u_2, \cdot), \cdot)_{L^2}$ we have, for $t \in [0, T]$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|f_1 - f_2\|_{L^2}^2 &\leq |(B(u_1 - u_2, f_1 + f_2), f_1 - f_2)_{L^2}| + 2|(g_1 - g_2, f_1 - f_2)| \\
&\leq C\|f_1 + f_2\|_{H^s}\|u_1 - u_2\|_{L^2}\|f_1 - f_2\|_{L^2} + 2\|g_1 - g_2\|_{\Sigma_0}\|f_1 - f_2\|_{L^2} \\
&\leq C\|f_1 + f_2\|_{\Sigma_s}\|u_1 - u_2\|_{\Sigma_0}\|f_1 - f_2\|_{L^2} + 2\|g_1 - g_2\|_{\Sigma_0}\|f_1 - f_2\|_{L^2}
\end{aligned}
$$

Where C depends on the embedding $H^{s-1} \hookrightarrow L^\infty$. Formally dividing by $\|f_1 - f_2\|_{L^2}$ and integrating the resulting inequality gives (3.29) or (3.30) depending on the choice of g_1 and g_2. Justifying this last step is straightforward. $\square$

We are now in a position to prove the main result.

Proof of Theorem 3.1. Fix $s > d/2 + 1$ and let C_3, C_4 be the constants in (3.18), (3.24) (from Lemmas 3.6 and 3.7) respectively. Fix $M > \|u_0\|_{H^s}$, then choose $T > 0$ so that

$$
\exp(C_4TM)\|u_0\|_{H^s}\left(\frac{C_3}{C_4}[\exp(C_4TM) - 1] + 1\right) \leq M.
$$

Let $u \in B_M(T)$ be a divergence-free function and let η be the solution of (3.23) for the flow u with initial data $\eta_0 = 0$ and forcing $g = -u$. Let v be the solution for initial data $v_0 = u_0$ with $g = 0$. Define $Su := \mathbb{P}[(\nabla\eta)^\top v + v]$, then by Lemmas 3.6 and 3.7,

$$
\|Su(t)\|_{H^s} \leq \exp(C_4tM)\|u_0\|_{H^s}\left(\frac{C_3}{C_4}[\exp(C_4tM) - 1] + 1\right) \leq M \quad (3.31)
$$

for all $t \in [0, T]$. Hence $S : B_M(T) \to B_M(T)$. Note that $Su(\cdot, 0) = u_0$ even if $u(\cdot, 0) \neq u_0$.

We next show that S is a contraction on $B_M(T)$ in the L^2 norm if T is sufficiently small. For u_1, $u_2 \in B_M(T)$ we construct v_i and η_i from u_i as above for $i = 1, 2$ with $v_1(\cdot, 0) = v_2(\cdot, 0) = u_0$. Now

$$
\begin{aligned}
\|Su_1 - Su_2\|_{L^2} &\leq C_a\|\eta_1 - \eta_2\|_{L^2} + C_b\|v_1 - v_2\|_{L^2} \\
&\leq (C_c\|v_1 + v_2\|_{\Sigma_s} + C_d\|\eta_1 + \eta_2\|_{\Sigma_s} + C_e)T\|u_1 - u_2\|_{\Sigma_0} \\
&\leq C(u_0, M, T)\|u_1 - u_2\|_{\Sigma_0},
\end{aligned} \quad (3.32)
$$

where $C_a, \ldots, C_e$ denote various constants arising from the application of Lemmas 3.6, 3.7 and 3.8. Keeping careful track of the constants shows that $C(u_0, M, T)$ is given by the formula

$$\begin{aligned} C(u_0, M, T) \coloneqq 2T \Bigg[\left(C_5 (C_3' M + 1) \|u_0\|_{H^s} + \frac{C_3' C_5 M}{C_4} \right) \exp(C_4 T M) \\ + C_3' M \left(\frac{1}{2} - \frac{C_5}{C_4} \right) \Bigg] \end{aligned} \tag{3.33}$$

Where C_3', C_4, C_5 are the constants from Lemmas 3.6, 3.7 and 3.8 respectively. Taking the supremum of (3.32) with respect to t and choosing $T > 0$ small enough, we see that S is a contraction in the required sense.

We conclude that S has a unique accumulation point u, in the closure of B_M with respect to $\|\cdot\|_{\Sigma_0}$. Since $B_M(T)$ is convex and closed in Σ_s, it is weakly closed, hence $u \in B_M(T)$ is a fixed point of S. A fixed point of S, along with associated back-to-labels map and virtual velocity, clearly give a solution to the Eulerian-Lagrangian formulation of the Euler equations with the required regularity. The contraction argument gives uniqueness in $B_M(T)$ and it remains to prove that we have uniqueness in $\Sigma_s(T)$.

Since S is a contraction on $B_M(\widetilde{T})$ for any $\widetilde{T} \in (0, T]$, we have by continuity of $\|u(t)\|_{H^s}$, that if u', A' and v' also satisfy (3.4–3.6) with $u' \in \Sigma_s(T)$, then $u(t) = u'(t)$ when

$$0 \le t \le \min(T, \inf\{r : \|u'(r)\|_{H^s} = M\}).$$

Now we know that for all $k \in \mathbb{N}$ there exists $T_k \le T$ such that S is a contraction on $B_{M+1/k}(T_k)$ and we may assume $T_k \to T$ as $k \to \infty$. By the previous observation, this means that u is the unique solution in $\Sigma_s(T - \varepsilon)$ for all $\varepsilon > 0$, hence by continuity u is the unique solution in Σ_s as required.

The proof that $u \in C^1([0, T]; H^{s-1})$ uses the same trick as Lemma 3.6 to save a spatial derivative (we have only shown that $\nabla \eta_t \in H^{s-2}$, which might otherwise limit the regularity of u). By definition $u = \mathbb{P}[(\nabla \eta)^\top v + v]$.

We use (3.22) from the proof of Lemma 3.6. Precisely we have

$$\begin{aligned}
&\frac{1}{h}\left\|u(t+h)-u(t)-h\mathbb{P}[(\nabla\eta(t))^\top\partial_t v(t)+\partial_t v(t)+(\nabla v(t))^\top\partial_t\eta(t)]\right\|_{H^{s-1}}\\
&\quad\le \frac{1}{2h}\left\|\mathbb{P}[(\nabla\eta(t+h)+\nabla\eta(t))^\top(v(t+h)-v(t)-h\partial_t v)]\right\|_{H^{s-1}}\\
&\quad\quad+\frac{1}{2h}\left\|\mathbb{P}[(\nabla v(t+h)+\nabla v(t))^\top(\eta(t+h)-\eta(t)-h\partial_t\eta)]\right\|_{H^{s-1}}\\
&\quad\quad+\frac{1}{2}\|\mathbb{P}[(\nabla\eta(t+h)-\nabla\eta(t))^\top\partial_t v(t)]\|_{H^{s-1}}\\
&\quad\quad+\frac{1}{2}\|\mathbb{P}[(\nabla v(t+h)-\nabla v(t))^\top\partial_t\eta(t)]\|_{H^{s-1}}\\
&\quad\quad+\frac{1}{h}\|v(t+h)-v(t)-h\partial_t v(t)\|_{H^{s-1}}.
\end{aligned}$$

Since H^{s-1} is an algebra and $\eta, v \in C([0,T];H^s)\cap C^1([0,T];H^{s-1})$, the right-hand side vanishes as $h\to 0$. Therefore $u\in C^1([0,T];H^{s-1})$ and

$$\partial_t u = \mathbb{P}[(\nabla\eta(t))^\top\partial_t v(t)+\partial_t v(t)(\nabla v(t))^\top\partial_t\eta(t)]. \qquad \square$$

3.4 An alternative iteration scheme

Here we exhibit an alternative proof of existence and uniqueness for (3.4–3.6), which is based on contractions with respect to A rather than u. Structuring the proof in this way follows Constantin (2000) more closely, however in the setting of Sobolev spaces it is less natural than the proof above because it relies on estimating compositions. As such we have only been able to adapt Constantin's method in this case for $s\in\mathbb{Z}$.

The extra technicality in this approach is contained in the following lemma, which is proved in the next section. We will denote the identity map on $\mathbb{T}^d$ by ι and use the correspondence between maps $\mathbb{T}^d\to\mathbb{R}^d$ and $\mathbb{T}^d\to\mathbb{T}^d$ without comment.

Lemma 3.9. *Let $s\in\mathbb{Z}$ with $s>\frac{d}{2}+1$ and fix $f,g\in H^s$. If $g+\iota$ is a volume preserving map then $f\circ(g+\iota)\in H^s$ and*

$$\|f\circ(g+\iota)\|_{H^s}\le C_6\|f\|_{H^s}(\|g\|_{H^s}+(2\pi)^d)^s \tag{3.34}$$

for some $C_6>0$ depending only on s and the constants from some Sobolev embeddings.

This allows us to write a second proof of existence and uniqueness of solutions in Σ_s for $s > d/2+1$ in the case $s \in \mathbb{Z}$.

Fix $u_0 \in H^s$ and $M > 0$ and suppose $\eta \in B_M(T)$ for some $T > 0$ such that $\eta(t) + \iota$ is volume-preserving for all $t \in [0, T]$. Define u and v via $v = u_0 \circ (\eta + \iota)$ and $u = \mathbb{P}[(\nabla\eta)^\top v + v]$. Construct η', the iterate of η by solving

$$\partial_t \eta' + (u \cdot \nabla)\eta' = -u, \ \eta'(x, 0) = 0.$$

By Lemmas 3.6, 3.7 and 3.9 we have

$$\|\eta'\|_{\Sigma_s} \le \frac{1}{C_4}\left[\exp(C_4 C_6 (C_3 M + 1)(M + (2\pi)^d)^s \|u_0\|_{H^s} T) - 1\right].$$

Hence for T small enough, we may assume $\eta' \in B_M(T)$ and since $\nabla \cdot u = 0$ we also have that $\eta' + \iota$ is volume preserving.

Now suppose that $\eta_1,\ \eta_2 \in B_M(T)$ and let $\eta_1',\ \eta_2'$ be the respective iterates then

$$\|\eta_1' - \eta_2'\|_{\Sigma_0} \le 2(C_5 M + 1)(C_3' M + (C_3' M + 1)C_{\mathrm{Lip}})T\|\eta_1 - \eta_2\|_{\Sigma_0},$$

by Lemmas 3.6 and 3.8. Here C_{Lip} is the Lipschitz constant of u_0. It follows that, for small enough T, this iteration procedure is a contraction on $B_M(T)$ in the L^2 norm. Existence and uniqueness of solutions now follows using the same steps as in the previous method.

3.5 Compositions in H^s

We now prove Lemma 3.9, which gives bounds on the compositions H^s functions with certain volume-preserving locally H^s functions where $s \in \mathbb{Z}$ with $s > \frac{d}{2}$.

To begin with we consider $g_i \in H^s$ and multi indices β_i with $|\beta_i| \in [1, s]$ for $i = 1, \dots, \ell$. We call $p \in [1, \infty]$ *admissible* for $(\beta_i)_{1 \le i \le \ell}$ if there exists a constant $C > 0$ independent of $(g_i)_{1 \le i \le \ell}$ such that

$$\left\| \prod_{i=1}^{\ell} \mathrm{D}^{\beta_i} g_i \right\|_{L^p} \le C \prod_{i=1}^{\ell} \|g_i\|_{H^s}. \tag{3.35}$$

Of course p is admissible if there exist $q_1, \dots, q_\ell \in [1, \infty)$ such that

$$\sum_{i=1}^{\ell} \frac{1}{q_i} = \frac{1}{p},$$

and $H^{s-|\beta_i|} \hookrightarrow L^{q_i}$ for each i, or $p = \infty$ and $q_i = \infty$ for all i. We may assume, without loss of generality that there are constants k_1 and k_2 with $0 \le k_1 \le k_2 \le \ell$ such that

$$\begin{cases} s - |\beta_i| \in [0, d/2) \text{ for } 1 \le i \le k_1 \\ s - |\beta_i| = d/2 \text{ for } k_1 + 1 \le i \le k_2 \\ s - |\beta_i| > d/2 \text{ for } k_2 + 1 \le i \le \ell \end{cases}$$

So we have

$$\left\| \prod_{i=1}^{k_1} D^{\beta_i} g_i \right\|_{L^p} \le C \prod_{i=1}^{k_1} \|g_i\|_{H^s}$$

for

$$\frac{1}{p} \in \left[\sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n}, \frac{k_1}{2} \right].$$

Moreover

$$\left\| \prod_{i=k_1+1}^{k_2} D^{\beta_i} g_i \right\|_{L^p} \le C \prod_{i=k_1+1}^{k_2} \|g_i\|_{H^s}$$

for $p \in [2, \infty)$. Lastly,

$$\left\| \prod_{i=k_2+1}^{\ell} D^{\beta_i} g_i \right\|_{L^\infty} \le C \prod_{i=k_2+1}^{\ell} \|g_i\|_{H^s}.$$

Combining these observations we see that p is admissible if

$$\frac{1}{p} \in \left(\sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n}, \frac{\ell}{2} \right]. \tag{3.36}$$

or if $k_1 = k_2$ then p is still admissible if

$$\frac{1}{p} = \sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n}, \tag{3.37}$$

furthermore $p = \infty$ is admissible if $k_1 = k_2 = 0$.

Note that if $p \in [1, \infty]$ is admissable and $f_i : \mathbb{T}^d \to \mathbb{R}^d$ are linear maps then we have (rather crudely)

$$\left\| \prod_{i=1}^{\ell} D^{\beta_i}(g_i + f_i) \right\|_{L^p} \leq C \prod_{i=1}^{\ell} \|g_i\|_{H^s} + \|f_i\|_{\mathrm{op}}(2\pi)^{n/q_i}. \tag{3.38}$$

In the proof of the lemma below, we will need the fact that if $s > \frac{d}{2}$ and $\sum_{i=1}^{\ell} |\beta_i| \leq s$ then $p = 2$ is admissible for $(\beta_i)_{1 \leq i \leq \ell}$. Furthermore, we will need to show that if $s > d/2 + 1$ then there exists an admissible $p > \frac{d}{s-\ell}$ and that $p = \infty$ is admissible if $s = \ell > d/2 + 1$.

For the first claim, note that if $k_1 = 0$ or $k_1 = 1$ then $p = 2$ is clearly admissible. Otherwise, if $1 < k_1 \leq \ell$ and $s > d/2$, we have the following calculation:

$$\sum_{i=1}^{k_1} n - 2(s - |\beta_i|) \leq k_1 n - 2k_1 s + 2s = (k_1 - 1)(n - 2s) + n < n \tag{3.39}$$

so $p = 2$ is admissible. For the second claim, observe that if $s > d/2 + 1$ then

$$\sum_{i=1}^{k_1} n - 2(s - |\beta_i|) < 2\sum_{i=1}^{k_1} |\beta_i| - 2k_1 \leq 2(s - k_1) - 2 \sum_{i=k_1+1}^{\ell} |\beta_i| \leq 2(s - \ell), \tag{3.40}$$

where the middle inequality uses the assumption that $\sum_{i=1}^{\ell} |\beta_i| \leq s$. Hence there exists an admissible value $p > \frac{d}{s-\ell}$, if $s - \ell > 0$. If $s = \ell$ then necessarily, $|\beta_i| = 1$ for $i = 1, \ldots, \ell$ hence $p = \infty$ is admissible by (3.37).

Lemma 3.10. *Let $s \in \mathbb{Z}$ with $s > \frac{d}{2} + 1$ and fix $f, g \in H^s$. Denote the identity map on $\mathbb{T}^d$ by ι. If $g + \iota$ is a volume preserving map then $f \circ (g + \iota) \in H^s(\mathbb{T}^d)$ and*

$$\|f \circ (g + \iota)\|_{H^s} \leq C\|f\|_{H^s}(\|g\|_{H^s} + (2\pi)^d)^s \tag{3.41}$$

for some $C > 0$ depending only on s and the constants from some Sobolev embeddings.

Proof. For each $k \in \mathbb{N}$, consider functions $f_k, g_k \in C^\infty(\mathbb{T}^d; \mathbb{R}^d)$ such that $f_k \to f$ in H^s and $g_k \to g$ in H^s. Without loss of generality we assume that $|| \det \nabla(g_k(x) + x)| - 1| < \frac{1}{k+1}$ holds uniformly in x.

Now by the chain and Leibniz rules, we see that for a multi-index γ with $|\gamma| \le s$, $D^\gamma(f_k \circ (g_k + \iota))$ is a (weighted) sum with summands of the form

$$((D^\alpha f_k) \circ (g_k + \iota)) \prod_{i=1}^{\ell} D^{\beta_i}(g_k^{r_i} + x_{r_i}), \tag{3.42}$$

where $\ell = |\alpha| \le |\gamma|$ and $\sum_{i=1}^{\ell} |\beta_i| = |\gamma|$. Here g_k^i denotes the ith vector component of g_k. We seek to bound terms of the form (3.42) in L^2 using the preceding observations.

Since $D^\alpha f_k \in H^{s-\ell}$ and $g_k + \iota$ is "almost volume preserving" it can be seen that $(D^\alpha f_k) \circ (g_k + \iota) \in L^q$ if

$$\frac{1}{q} \in \left(\frac{1}{2} - \frac{s-\ell}{n}, \frac{1}{2}\right]$$

with $s - \ell \in (0, n/2]$ or

$$\frac{1}{q} = \frac{1}{2} - \frac{s-\ell}{n}$$

when $s - \ell \in (0, n/2)$. Of course, if $s - \ell > d/2$ then $D^\alpha f_k \in L^\infty$.

To bound (3.42) in L^2 therefore, we need to check that there is an admissible p such that,

$$\frac{1}{p} \in \left[0, \frac{s-\ell}{n}\right).$$

and that $p = \infty$ is admissible if $s = \ell$. This follows from the claims we proved before the statement of the lemma.

Now we see that

$$\|f_k \circ (g_k + \iota)\|_{H^s} \le C\sqrt{1 + 1/k}\, \|f_k\|_{H^s}(\|g_k\|_{H^s} + (2\pi)^d)^s$$

where C depends only on Sobolev embeddings and some combinatorics. Since f_k and g_k converge we may assume that $f_k \circ (g_k + \iota)$ converges weakly in H^s. Thus the lemma is proved if we can show that $f_k \circ (g_k + \iota) \to f \circ (g + \iota)$ in L^2 for example. This is indeed the case:

$$\begin{aligned}
&\|f \circ (g + \iota) - f_k \circ (g_k + \iota)\|_{L^2} \\
&\qquad \le \|f \circ (g + \iota) - f \circ (g_k + \iota)\|_{L^2} + \|f \circ (g_k + \iota) - f_k \circ (g_k + \iota)\|_{L^2} \\
&\qquad \le C_{\text{Lip}}\|g - g_k\|_{L^2} + \sqrt{1 + 1/k}\, \|f - f_k\|_{L^2},
\end{aligned}$$

where we make use of the fact that $f \in H^s$ is Lipschitz since $s > d/2 + 1$

and denote by C_{Lip} the Lipschitz constant of f. $\square$

3.6 Remarks on the diffusive case

In this section we discuss the difficulties that arise when seeking an analogous reformulation of the Navier–Stokes equations. Recall that these are obtained by adding a diffusion term:

$$u_t + (u \cdot \nabla)u - \Delta u + \nabla p = f. \tag{3.43}$$

Constantin has discussed the following Eulerian-Lagrangian version of the Navier–Stokes equations

$$\partial_t A + (u \cdot \nabla)A - \nu \Delta A = 0, \tag{3.44}$$

$$u = \mathbb{P}((\nabla A)^\top v), \tag{3.45}$$

$$\partial_t v + (u \cdot \nabla)v - \nu \Delta v = 2\nu C \nabla v. \tag{3.46}$$

Notice that in this system A is a diffusive analogue of the back-to-labels map. This makes it much more difficult to form a useful interpretation of this new form for the Navier-Stokes equations. The other main difference from (3.4–3.6) is the introduction of a matrix of coefficients C in the equation for v. This matrix depends on the derivatives of A and $(\nabla A)^{-1}$ and as such is an active part of the equation in some sense (i.e. the behaviour of C is something new to consider).

Another difficulty with (3.44) is that, unlike in the case of the Euler equations, the divergence free property of u is not enough to ensure the invertibility of A (which is needed to calculate C) hence in general one needs to keep resetting this system (after certain intervals of time) in order for solutions to remain solutions of the Eulerian form of the Navier-Stokes equations. For details of this resetting argument see Constantin (2008).

One might expect that a formulation more akin to (3.4–3.6) is meaningful in the presence of diffusion. More precisely the results discussed in the following few paragraphs indicate that particle trajectories and in particular the back-to-labels map are sensible concepts even for weak solutions.

Foias, Guillopé, and Témam (1985) showed that a Leray–Hopf weak solutions of the Navier–Stokes equations on a 3D bounded domain admits a volume-preserving Lagrangian trajectory map $\phi \in L^\infty(\Omega; C([0,T];\Omega))$. Furthermore, DiPerna and Lions (1989) showed that if only

$$b \in L^1(0,T; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)) \text{ and } \nabla \cdot b \in L^1(0,T; L^\infty)$$

with the decay condition

$$\frac{|b(x,t)|}{1+|x|} \in L^1(0,T; L^1(\mathbb{R}^d)) + L^1(0,T; L^\infty(\mathbb{R}^d))$$

then there is a unique solution X of

$$\frac{\mathrm{d}}{\mathrm{d}t} X(a,t) = b(t, X(a,t)), X(0,x) = x$$

in the sense of so-called renormalised solutions of the transport equation

$$\partial_t g - (b \cdot \nabla) g = 0.$$

That is a function g, such that for any bounded $\beta \in C^1(\mathbb{R})$ that vanishes near zero,

$$\partial_t \beta(g) - (b \cdot \nabla)\beta(g) = 0$$

holds in the sense of distributions.

In particular the results of DiPerna and Lions imply the uniqueness of the trajectory map of Foias et al. For a brief discussion of the equivalence between these notions of a trajectory map, see Lions (1998).

As we discussed in Chapter 2, Robinson and Sadowski (2009b) proved existence and uniqueness of almost every particle trajectory for suitable weak solutions to the Navier–Stokes equations in three-dimensions.

Given this background it seems reasonable to investigate whether the Navier–Stokes equations has a meaningful Eulerian-Lagrangian formulation, in which the back-to-labels map is the honest back-to-labels map, in the sense that it is not diffused. In the next subsections we will follow the derivation of the Weber formula in the presence of the diffusion term and give preliminary comments on the analysis of that formulation.

3.6.1 A diffusive version of the Weber formula

Deriving a diffusive Weber formula

We now show that if u satisfies the Navier–Stokes equations then u solves a system of equations similar to (3.4–3.6). We begin with a sufficiently smooth solution of the Navier–Stokes equations.

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0 \tag{3.47}$$

with

$$\nabla \cdot u = 0 \tag{3.48}$$

for some fixed $\nu \geq 0$. In the following calculations we will not assume that $\nu = 1$, so that we can keep track of the parameter in the resulting formulation.

As for the Euler equations, we next consider the trajectory map X defined by the equation

$$\partial_t X(a,t) = u(X(a,t),t) \tag{3.49}$$

for all $t \in [0,T]$ and all $a \in \Omega$. The initial condition is $X(a,0) = a$ for all a. Taking two time derivatives yields

$$\begin{aligned}\partial_{tt} X(a,t) &= \partial_t u(X(a,t),t) + \nabla u(X(a,t),t)\partial_t X(a,t) \\ &= -\nabla p(X(a,t),t) + \nu\Delta u(X(a,t),t).\end{aligned}$$

Letting $\tilde{p}(a,t) := p(X(a,t),t)$ and multiplying this equation by $(\nabla X)^\top$ gives

$$\left[\partial_{a_i} X^k \partial_{tt} X^k\right]_i = -\nabla\tilde{p}(a,t) + \nu(\nabla X)^\top \Delta u(X(a,t),t)$$

where the left hand side should be interpreted as a column vector (with the ith term given as a sum over k). Taking a time derivative out of the left hand side yields

$$\partial_t \left[\partial_{a_i} X^k \partial_t X^k\right]_i = -\nabla\left(\tilde{p} - \frac{1}{2}|\partial_t X|^2\right) + \nu(\nabla X)^\top \Delta u.$$

Integrating with respect to time and using (3.49) and the initial data for

X, we obtain

$$\left[\partial_{a_i} X^k \partial_t X^k\right]_i = u(a,0) - \int_0^t \nabla\left(\tilde{p}(a,s) - \frac{1}{2}|\partial_t X|^2(a,s)\right) ds \qquad (3.50)$$

$$+ \nu \int_0^t (\nabla X)^\top \Delta u(X(a,s),s)\mathrm{d}s$$

$$= u(a,0) - \nabla\tilde{n}(a,t) + \nu \int_0^t (\nabla X)^\top \Delta u(X(a,s),s)\mathrm{d}s. \qquad (3.51)$$

where $\tilde{n}(a,t) = \int_0^t \tilde{p}(a,s) - \frac{1}{2}|\partial_t X|^2(a,s)ds$.

As before, we denote the back-to-labels map by A. Multiplying (3.51) on the left by $(\nabla A)^\top(X(a,t),t)$ and evaluating at $a = A(x,t)$ gives

$$\begin{aligned}\partial_t X(A(x,t),t) = &(\nabla A)^\top u(A(x,t),0) \\ &+ \nu(\nabla A)^\top \int_0^t (\nabla X)^\top (A(x,t),s)\Delta(u + \nabla\tilde{q})\big|_{(X(A(x,t),s),s)}\mathrm{d}s \\ &- (\nabla A)^\top \nabla\left(\tilde{n}(a,t) + \nu\int_0^t \Delta\tilde{q}(X(a,s),s)\mathrm{d}s)\right)\Big|_{a=A(x,t)}, \end{aligned} \qquad (3.52)$$

where we have added and subtracted an integral involving a function $\tilde{q}$ that we will define shortly. By the chain-rule, the last term can be replaced by ∇q for a (scalar) function q. Moreover the left-hand side is simply $u(x,t)$. We have therefore written u in the following form:

$$u(x,t) = (\nabla A)^\top(x,t)v(x,t) - \nabla q(x,t) \qquad (3.53)$$

with

$$\begin{aligned}\partial_t v = &\nabla u \cdot \partial_t A + \nu((\nabla A)^\top)^{-1}\Delta(u(x,t) + \nabla\tilde{q}) \\ &+ \nu\nabla_a \int_0^t (\nabla X)^\top(a,s)\Delta(u+\tilde{q})\big|_{(X(a,s),s)}\mathrm{d}s\Big|_{A(x,t)} \cdot \partial_t A\end{aligned} \qquad (3.54)$$

and

$$\nabla v = \nabla u \cdot \nabla A + \nu\nabla_a \int_0^t (\nabla X)^\top(a,s)\Delta u(X(a,s),s)\mathrm{d}s\Big|_{A(x,t)} \cdot \nabla A. \qquad (3.55)$$

Since $\partial_t A + (u \cdot \nabla)A = 0$, (3.54) and (3.55) together imply that v satisfies

$$\partial_t v + (u \cdot \nabla)v - \nu((\nabla A)^\top)^{-1}\Delta(u + \nabla\tilde{q}) = 0 \tag{3.56}$$

where all terms are evaluated at (x,t).

For this argument to be considered a converse to what follows we would prefer v to satisfy

$$\partial_t v + (u \cdot \nabla)v - \nu((\nabla A)^\top)^{-1}\Delta\left[(\nabla A)^\top v\right] = 0. \tag{3.57}$$

To justify this it suffices, by (3.53), to find $\tilde{q}$ such that $\nabla q = \nabla\tilde{q}$. By (3.52), it is enough that

$$\tilde{q}(x,t) = \tilde{n}(A(x,t),t) + \nu\int_0^t \Delta\tilde{q}(X(A(x,t),s),s)\mathrm{d}s, \tag{3.58}$$

or, equivalently

$$\tilde{q}(X(a,t),t) = \tilde{n}(a,t) + \nu\int_0^t \Delta\tilde{q}(X(a,s),s)\mathrm{d}s. \tag{3.59}$$

This is of course a heat equation for $\tilde{q} \circ X$ with forcing $\partial_t\tilde{n}$, so the (formal) derivation of (3.57) from (3.56) is justified under some mild conditions on n and the domain Ω (see, for example, Evans (2010)).

We have shown that a sufficiently regular solution to the Navier–Stokes equations can be recovered from a virtual velocity satisfying (3.57), using the Weber formula. We next prove the converse derivation, i.e. that a sufficiently smooth solution to this new Eulerian-Lagrangian system is a solution of the classical system.

The converse derivation

We consider the following Eulerian-Lagrangian system

$$\partial_t A + (u \cdot \nabla)A = 0 \tag{3.60}$$

$$u = (\nabla A)^\top v - \nabla q \tag{3.61}$$

$$\partial_t v + (u \cdot \nabla)v - \nu((\nabla A)^\top)^{-1}\Delta\left[(\nabla A)^\top v\right] = 0 \tag{3.62}$$

with initial conditions

$$A(x,0) = x,\ v(x,0) = u_0(x),\ \forall x \in \Omega. \tag{3.63}$$

Here q is determined (up to addition of a constant) by the incompressibility constraint on u.

We now set about proving formally that if u, A, v satisfy (3.60- 3.62) and (3.63) then u is a solution to the Navier-Stokes equations with the same initial data. If we let $\nu = 0$ then we instead obtain the Euler equations.

To begin with, we recall the notation $D_t \coloneqq \partial_t + (u \cdot \nabla)$ and the commutation relation

$$D_t \nabla f = \nabla D_t f - (\nabla u)^\top \nabla f. \tag{3.64}$$

Applying D_t to (3.61) yields

$$\partial_t u + (u \cdot \nabla) u = (\partial_t (\nabla A)^\top) v + (\nabla A)^\top \partial_t v + (u \cdot \nabla)((\nabla A)^\top v) - D_t \nabla q.$$

By (3.62), this is

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u = (\partial_t (\nabla A)^\top) v + \nu \Delta \left[(\nabla A)^\top v \right] - (\nabla A)^\top (u \cdot \nabla) v \\ + (u \cdot \nabla)((\nabla A)^\top v) - D_t \nabla q. \end{aligned} \tag{3.65}$$

One can check, by considering components, that

$$(u \cdot \nabla)((\nabla A)^\top v) = (\nabla A)^\top (u \cdot \nabla) v + (\nabla [(u \cdot \nabla) A])^\top v - (\nabla u)^\top (\nabla A)^\top v.$$

Moreover, by (3.60), we easily obtain that

$$\partial_t (\nabla A)^\top + (\nabla (u \cdot \nabla) A)^\top = 0.$$

Hence (3.65) reduces to

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= \nu \Delta (\nabla A)^\top v - (\nabla u)^\top (\nabla A)^\top v - D_t \nabla q \\ &= \nu \Delta u + \nu \Delta \nabla q - (\nabla u)^\top (\nabla A)^\top v - D_t \nabla q. \end{aligned}$$

Now using the commutation relation (3.64) we have

$$\begin{aligned}(\nabla u)^\top(\nabla A)^\top v + D_t\nabla q &= (\nabla u)^\top(\nabla A)^\top v + \nabla D_t q - (\nabla u)^\top\nabla q\\ &= (\nabla u)^\top u + \nabla D_t q = \nabla\left(\frac{1}{2}|u|^2 + D_t q\right).\end{aligned}$$

Finally, setting $p = \frac{1}{2}|u|^2 + D_t q - \nu\Delta q$, we have shown that

$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0$$

i.e. u solves the Navier-Stokes (Euler) equations.

3.6.2 Remarks on possible analysis

It is not immediately clear whether arguments similar to those in this chapter are applicable to the system (3.60–3.62). Indeed, the "diffusion" term in (3.62) does not seem amenable to any familiar energy estimates.

As an exercise, it might be interesting to investigate certain simplifications of this system. For example, if we replace $(\nabla A)^\top$ in the equation for v with a fixed family of invertible matrices $M(x,t)$ or $M(x)$, we obtain the following system:

$$\partial_t A + (u\cdot\nabla)A = 0, \tag{3.66}$$

$$u = \mathbb{P}((\nabla A)^\top v), \tag{3.67}$$

$$\partial_t v + (u\cdot\nabla)v - M^{-1}\Delta(Mv) = 0. \tag{3.68}$$

For a general family of matrices M, this does not appear to be much simpler. Let us therefore assume, in addition, that M is a family of orthogonal matrices. In that case

$$-(M^\top\Delta(Mv), v)_{H^s} \geq \|Mv\|^2_{H^{s+1}},$$

so a solution v of (3.68) belongs to $L^\infty(0,T;H^s)$ and can be estimated using the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^s} + \|\nabla(Mv)\|^2_{H^s} \leq c\|u\|_{H^s}\|v\|^2_{H^s},$$

if $s > 1 + d/2$. It should be possible to follow the proof of Theorem 3.1 to obtain local well-posedness for (3.66–3.68) in $C(0,T;H^s(\mathbb{T}^3))$, for

$s > 1 + d/2$, in the same way.

Moreover, in addition to the estimate in Lemma 3.7, we can expect estimates on $v \in L^2(0, T; H^{s+1})$, in terms of $\|u\|_{L^1(0,T;H^s)}$. However, it is not immediately clear whether this leads to improved estimates on u. Indeed, in the proof of Lemma 3.6, our estimates were based on the observation that

$$\|\mathbb{P}[(\nabla\eta)^\top v]\|_{H^s} = \|\mathbb{P}[(\nabla v)^\top \eta]\|_{H^s} \leq \|v\|_{H^s}\|\eta\|_{H^s},$$

for $\eta,\ v \in H^s$ and $s > 1 + d/2$. It is not obvious why estimates on one additional derivative of v (but not η) should lead to higher-order estimates on u.

In Chapter 6 we will study the so-called magnetization variables formulation of the Navier–Stokes equations (previously discussed in Montgomery-Smith and Pokorný (2001)), and a new model for that system for which we can prove global well-posedness in $H^{1/2}(\mathbb{T}^3)$. Given the formulations in this section, it is not difficult to derive the formulation in magnetization variables. Indeed, if we consider the equation satisfied by $w := (\nabla A)^\top v$ in (3.60–3.62), we obtain (using the commutation relation (3.64))

$$\partial_t w + (u \cdot \nabla)w + (\nabla u)^\top w - \Delta w = 0 \tag{3.69}$$

with

$$u = \mathbb{P}w.$$

This is the magnetization variables formulation of the Navier–Stokes equations.

Chapter 4

Well-posedness results for the Burgers equations in L^p

4.1 Introduction

At the end of the previous chapter, we briefly introduced the magnetization-variables formulation of the Navier–Stokes equations (3.69). That formulation gives rise to a particular model system sharing several features with the diffusive Burgers equations. The magnetization variables and the model system are the subject of Chapter 6. Partly as a preparation for the analysis there, in this chapter and the next we will give several existence and uniqueness results for the diffusive Burgers equations in dimensions $d = 2$, 3.

The equations, for a fixed viscosity $\nu > 0$ and initial data u_0, comprise the following system:

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u = 0, \tag{4.1}$$

$$u(0) = u_0. \tag{4.2}$$

As with the Navier–Stokes equations, it suffices to consider the case $\nu = 1$, by virtue of the rescaling $\widetilde{u}(x,t) := \nu u(x, \nu t)$.

Using the Burgers equations as a model in the study of the Navier–Stokes equations sometimes seems to be part of the folklore of the theory of the latter system. However, we are motivated to give a careful treatment of the Burgers equations, here and in Pooley and Robinson (2016a), because we have not found much analysis in the literature that is directly comparable to well-known analyses of Navier–Stokes equations.

It should be noted that, despite the global nature of the results we obtain, by virtue of a maximum principle, the absence of an incompressibility constraint means that some of the familiar arguments do not translate directly to the Burgers equations. Indeed we do not obtain the L^2 estimates necessary to construct weak solutions from L^2 data and we must also consider the non-conservation of momentum, mentioned previously.

We begin with some brief comments on several relevant methods from the literature to motivate our discussion here.

A maximum principle for classical solutions of the Burgers equations was proved by Kiselev and Ladyzhenskaya (1957). A simplified version of this result with zero forcing plays a key role in our argument, so we reproduce the proof here.

Lemma 4.1. *If u is a classical solution of the Burgers equations (4.1), either in $\Omega = \mathbb{T}^d$ or a bounded domain $\Omega \subset \mathbb{R}^d$ with $u\big|_{\partial\Omega} = 0$, on a time interval $[a, b]$ then*

$$\sup_{t\in[a,b]} \|u(t)\|_{L^\infty} \le \|u(a)\|_{L^\infty}. \tag{4.3}$$

Proof. Fix $\alpha > 0$ and let $v(t, x) := \mathrm{e}^{-\alpha t} u(x, t)$ for all $x \in \Omega$. Then $|v|^2$ satisfies the equation

$$\frac{\partial}{\partial t}|v|^2 + 2\alpha|v|^2 + u \cdot \nabla|v|^2 - 2v \cdot \Delta v = 0. \tag{4.4}$$

Since $2v \cdot \Delta v = \Delta|v|^2 - 2|\nabla v|^2$ we see that if $|v|^2$ has a local maximum at $(x, t) \in (a, b] \times \Omega$ then the left-hand side of (4.4) is positive unless $|v(x, t)| = 0$. Hence

$$\|u(t)\|_{L^\infty} \le \mathrm{e}^{\alpha t}\|u(a)\|_{L^\infty}.$$

Now (4.3) follows because $\alpha > 0$ was arbitrary. □

In the discussion of well-posedness for (4.1) in Kiselev and Ladyzhenskaya (1957) the maximum principle is used with approximations obtained by considering discrete times and replacing the time derivatives by difference quotients. Unfortunately one of the steps there is incorrect. In the MathSciNet review, R. Finn relates a comment by L. Nirenberg that there is a flaw in the compactness argument given on pp. 675. This error appears to be fatal.

Another well known approach comes by analogy with the Burgers equations in one dimension, namely the Cole–Hopf transformation, which gives analytic solutions by reducing the problem to solving a heat equation. Unfortunately this can only give gradient solutions, and since we wish to draw comparisons with the classical equations of fluid mechanics this is a significant drawback.

There is a theorem by Ladyzhenskaya, Solonnikov, and Ural'ceva (1968) (Chapter VII, Theorem 7.1) giving local well–posedness for a certain class of quasi-linear parabolic problems that includes (4.1). In that theorem the data and solutions are taken to have spatial Hölder regularity[1] at least $C^{2,\alpha}$ for some $\alpha \in (0,1)$. It is likely that a consequence is global well–posedness in these spaces, but this is not stated. A brief sketch of the proof is given, but it is quite different from any familiar method used for the Navier–Stokes equations. Moreover there is no discussion of solutions gaining regularity that we will demonstrate (see Lemma 4.5).

In this chapter, we will prove several global well-posedness for the Burgers equations in various function spaces and domains in 2 and 3 dimensions. We begin with global existence results in H^1 and use these to construct solutions in L^p for $p > d$ for (smooth) bounded domains $\Omega \subset \mathbb{R}^d$ or the whole space.

We note that the H^1 theory of the Burgers equations was previously discussed by Heywood and Xie (1997). Along similar lines, we will give give a careful proof of global well-posedness in H^1 before proving new global well-posedness results for initial data in L^p.

It would also be natural to consider the case of initial data in L^∞, because of the maximum principle. For bounded domains, at least, global existence and uniqueness will follow from our analysis in L^p. In this chapter, our well–posedness arguments in L^p ($p \in (d, \infty)$) on the whole space rely on the additional decay $u_0 \in L^p \cap L^2$. It seems reasonable to expect that this may be extended to $L^\infty \cap L^2(\mathbb{R}^d)$, however a different approach may be needed if we only take $u_0 \in L^\infty(\mathbb{R}^d)$.

In the spatial dimension $d = 3$ we can extend this to the critical space $H^{1/2}$, at least in the case of periodic boundary conditions. This will be discussed in the next chapter, which is based on a paper by Pooley and

[1] The spaces in which solutions are found are actually defined by the existence and Hölder continuity (with certain exponent) of the mixed derivatives $D_t^r D_x^s$ for $2r + s < 2 + \alpha$, where D_x^s is any spatial derivative of order s.

Robinson (2016a).

In any of the domains we will discuss in this chapter or the next, we call u a weak solution of the Burgers equations if

$$u \in C_w([0,T), L^2) \cap L^2(0,T; H_0^1)$$

and for any $\phi \in C_c^\infty([0,T) \times \Omega)$ it satisfies

$$(u(t), \phi(t))_{L^2} + \int_0^t (\nabla u, \nabla \phi)\,\mathrm{d}s = (u(0), \phi(0)) + \int_0^t (u(s), \partial_t \phi(s))\,\mathrm{d}s \\ - \int_0^t ((u \cdot \nabla)u, \phi(s))\,\mathrm{d}s \quad (4.5)$$

for all $t \in [0,T)$.

4.2 Uniqueness results

We begin with a proof of uniqueness that applies to the solutions constructed in this chapter, starting with the three-dimensional case.

Lemma 4.2. *If $\Omega = \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain and u, $v \in C_w([0,T); L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ are weak solutions of (4.5) and additionally u, $v \in L^\infty(0,T; L^p)$ for some $p > 3$ and $\partial_t u, \partial_t v \in L^2(0,T; H^{-1})$. If u and v correspond to the same initial data $u_0 \in L^p \cap L^2$ then $u = v$.*

Proof. Let $r = \min(p, 6)$ and fix $t \in (0,T)$. Let $\phi_n \in C_c^\infty([0,T) \times \Omega)$ be a sequence such that

$$\phi_n \to u - v \ \text{ in } L^2(0,t; H^1)$$

and

$$\partial_t \phi_n \to \partial_t(u - v) \ \text{ in } L^2(0,t; H^{-1}),$$

(see Lemma 1.12). Moreover we may assume that u, $v \in C([0,T]; L^2)$ and ϕ_n converges to $u - v$ in L^2 uniformly on $[0,t]$.

Using ϕ_n as a test function in the difference of the equations (4.5) satisfied by u and v, the linear terms converge as follows:

$$([u-v](t), \phi_n(t))_{L^2} - ([u-v](0), \phi_n(0))_{L^2} \to \|[u-v](t)\|_{L^2}^2,$$

$$\int_0^t (\nabla(u-v)(s), \nabla \phi_n(s))\,\mathrm{d}s \to \int_0^t \|\nabla(u-v)(s)\|_{L^2}^2\,\mathrm{d}s,$$

and (using Lemma 1.12 again)

$$\int_0^t ([u-v](s), \partial_t \phi_n(s))\,\mathrm{d}s \to \int_0^t \langle (u-v)(s), \partial_t (u-v)(s)\rangle_{H_0^1 \times H^{-1}}\,\mathrm{d}s$$
$$= \frac{1}{2}\|(u-v)(t)\|_{L^2}^2.$$

For one of the nonlinear terms we have

$$\left|\int_0^t (u\cdot\nabla(u-v), \phi_n(s))\,\mathrm{d}s\right| \le \int_0^t \|u\|_{L^r}\|\nabla(u-v)\|_{L^2}\|\phi_n(s)\|_{L^{2r/(r-2)}}\,\mathrm{d}s$$
$$\le \int_0^t \|u\|_{L^r}\|\nabla(u-v)\|_{L^2}\|\phi_n\|_{L^6}^{3/r}\|\phi_n\|_{L^2}^{(r-3)/r}\,\mathrm{d}s$$
$$\le \frac{1}{4}\int_0^t \|\nabla(u-v)\|_{L^2}^{2r/(r+3)}\|\phi_n\|_{H^1}^{6/(r+3)}\,\mathrm{d}s + c\int_0^t \|u\|_{L^r}^{2r/(r-3)}\|\phi_n\|_{L^2}^2\,\mathrm{d}s.$$

The limit of the right-hand side as $n\to\infty$ is at most

$$\frac{1}{4}\int_0^t \|(u-v)(s)\|_{H^1}^2\,\mathrm{d}s + c\int_0^t \|u\|_{L^r}^{2r/(r-3)}\|u-v\|_{L^2}^2\,\mathrm{d}s$$
$$\le \frac{1}{4}\int_0^t \|\nabla(u-v)\|_{L^2}^2\,\mathrm{d}s + c\int_0^t \left(\|u\|_{L^r}^{2r/(r-3)}+1\right)\|u-v\|_{L^2}^2\,\mathrm{d}s.$$

To estimate the other nonlinear term we first integrate by parts to remove the derivatives from v as follows:

$$\left|\int_0^t ((u-v)\cdot\nabla v, \phi_n(s))\,\mathrm{d}s\right|$$
$$= \left|\int_0^t ((u-v)\cdot\nabla\phi_n(s), v) + (\nabla\cdot(u-v)v, \phi_n(s))\,\mathrm{d}s\right|$$
$$\le c\int_0^t \left(\|u-v\|_{L^{2r/r-2}}\|\nabla\phi_n\|_{L^2} + \|\phi_n\|_{L^{2r/r-2}}\|\nabla(u-v)\|_{L^2}\right)\|v\|_{L^r}\,\mathrm{d}s.$$

Proceeding as above, we obtain

$$\limsup_{n\to\infty}\left|\int_0^t ((u-v)\cdot\nabla v, \phi_n(s))\,\mathrm{d}s\right|$$
$$\le \frac{1}{4}\int_0^t \|\nabla(u-v)\|_{L^2}^2\,\mathrm{d}s + c\int_0^t (\|v\|_{L^r}^{2r/(r-3)}+1)\|u-v\|_{L^2}^2\,\mathrm{d}s.$$

Combining all of the above estimates yields

$$\|(u-v)(t)\|_{L^2}^2 \leq c\left(\|v\|_{L^\infty(0,T;L^r)}^{2r/(r-3)} + \|u\|_{L^\infty(0,T;L^r)}^{2r/(r-3)} + 1\right)\int_0^t \|(u-v)(s)\|_{L^2}^2\,\mathrm{d}s.$$

Solving this integro-differential inequality, using the fact that $u(0) = v(0)$ it follows that $u = v$. $\square$

We now prove the corresponding result in two dimensions.

Lemma 4.3. *If $\Omega = \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and u, $v \in C_w(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ are weak solutions of (4.5) and additionally u, $v \in L^\infty(0,T;L^p)$ for some $p > 2$ and $\partial_t u, \partial_t v \in L^2(0,T;H^{-1})$. If u and v correspond to the same initial data $u_0 \in L^p \cap L^2$ then $u = v$.*

Proof. Here we give only a sketch of the proof since the details can easily be adapted from the previous lemma. In particular, we will proceed formally with $\phi = u - v$ as a "test function", in order to illustrate the estimates that can be made.

The linear terms do not require additional comment. For the nonlinear term it suffices to consider $u \cdot \nabla(u-v)$. For any $r > 2p/(p-2)$ we use the embedding $H_0^1 \hookrightarrow L^r$ to obtain

$$\begin{aligned}
\left|\int_0^t (u\cdot\nabla(u-v), u-v)\,\mathrm{d}s\right| &\leq \int_0^t \|\nabla(u-v)\|_{L^2}\|u\|_{L^p}\|u-v\|_{L^{2p/(p-2)}} \\
&\leq \|\nabla(u-v)\|_{L^2}\|u\|_{L^p}\|u-v\|_{L^2}^{1-\frac{2r}{(r-2)p}}\|u-v\|_{L^r}^{\frac{2r}{(r-2)p}} \\
&\leq c\int_0^t \|u-v\|_{H^1}^{\frac{2r+(r-2)p}{(r-2)p}}\|u\|_{L^p}\|u-v\|_{L^2}^{1-\frac{2r}{(r-2)p}} \\
&\leq \frac{1}{4}\int_0^t \|\nabla(u-v)\|_{L^2}^2 + c\int_0^t (\|u\|_{L^p}^{\frac{2(r-2)p}{(r-2)p-2r}} + 1)\|u-v\|_{L^2}^2.
\end{aligned}$$

It follows that $u - v$ satisfies the integro-differential inequality

$$\|u-v(t)\|_{L^2}^2 \leq c\left(\|v\|_{L^\infty(0,T;L^p)}^{\frac{2(r-2)p}{(r-2)p-2r}} + \|u\|_{L^\infty(0,T;L^p)}^{\frac{2(r-2)p}{(r-2)p-2r}} + 1\right)\int_0^t \|u-v\|_{L^2}^2\,\mathrm{d}s.$$

The rest of the argument and the full justification are analogous to the proof of Lemma 4.2. $\square$

4.3 Global existence in H^1: bounded domains

In this section we will prove global existence for the Burgers equations in $H^1(\Omega)$ for smooth bounded domains Ω in two or three dimensions. We first prove local existence and smoothness of the solution after the initial time, then extend the solution globally using the maximum principle and the uniqueness result in the previous section.

In the next chapter we will use similar results for $u_0 \in H^1(\mathbb{T}^3)$, without giving detailed proofs. A full discussion of well-posedness for the Burgers equations in $H^1(\mathbb{T}^3)$ can be found in Pooley and Robinson (2016a).

4.3.1 Local existence

Lemma 4.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. Given $u_0 \in H^1_0(\Omega)$ there exists $T > 0$ and a weak solution u of the Burgers equations on $[0,T)$ with the additional regularity*

$$u \in C([0,T); H^1_0) \cap L^2(0,T; H^2).$$

Proof. Fix an orthonormal basis $\{\xi_k\}_{k=1}^\infty$ of $L^2(\Omega)$, consisting of eigenfunctions of the Laplacian:

$$-\Delta\xi_k = \lambda_k\xi_k,\ \xi_k \in C_c^\infty(\Omega)$$

for some $\lambda_k > 0$, and let $\mathcal{P}_n$ denote the projection onto $\langle\xi_1,\dots,\xi_n\rangle$. Now for some $T_n > 0$ there exists a classical solution $u_n \in C^1([0,T_n]; C_c^\infty)$ (with $u_n = \mathcal{P}u_n$) of the following quadratic system of ODEs

$$\partial_t u_n - \Delta u_n = -\mathcal{P}_n[(u_n \cdot \nabla)u_n],\ u_n(0) = \mathcal{P}_n u_0. \tag{4.6}$$

Integrating this system against $-\Delta u_n$ yields the estimate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n\|_{L^2}^2 + \|\Delta u_n\|_{L^2}^2 &\le \|u_n\|_{L^6}\|\nabla u_n\|_{L^3}\|\Delta u_n\|_{L^2} \\ &\le \|u_n\|_{H^1_0}^{3/2}\|u_n\|_{H^2}^{3/2} \le c\|\nabla u_n\|_{L^2}^6 + \frac{1}{2}\|\Delta u_n\|_{L^2}^2, \end{aligned} \tag{4.7}$$

where we have used Hölder exponents $(3,6,2)$, interpolated L^3 between L^2 and L^6, and the embedding $H^1_0 \hookrightarrow L^6$, which holds in two or three dimensions. The last inequality follows by the inequalities of Young and

Poincaré.

It follows that $X := \|\nabla u_n\|_{L^2}^2$, satisfies a differential inequality of the form

$$\frac{\mathrm{d}X}{\mathrm{d}t} \leq cX^3,$$

which implies that, for some $c > 0$, X is bounded on a time interval $[0, S]$ for any

$$S < \frac{1}{c|X(0)|}$$

with a bound depending only on $|X_0|$. Hence we may assume that for each n,

$$T_n > \frac{1}{2c\|\nabla u_n(0)\|_{L^2}^2} \geq \frac{1}{2c\|\nabla u_0\|^2}$$

and choose

$$0 < T \leq \frac{1}{2c\|\nabla u_0\|^2}.$$

By (4.7) the approximants u_n are bounded in $L^\infty(0, T; H_0^1) \cap L^2(0, T; H^2)$ independent of n. Moreover, integrating (4.6) against $\psi \in L^2(\Omega)$ yields:

$$\begin{aligned}(\partial_t u_n(t), \psi)_{L^2} &= (\Delta u_n(t), \psi)_{L^2} - ((u_n \cdot \nabla) u_n(t), \psi)_{L^2} \\ &\leq \left(\|\Delta u_n(t)\|_{L^2} + c\|u_n(t)\|_{H_0^1}^{3/2} \|u_n(t)\|_{H^2}^{1/2} \right) \|\psi\|_{L^2},\end{aligned}$$

so $\partial_t u_n$ is bounded in $L^2(0, T; L^2)$ independent of n. After applying the Aubin-Lions lemma, we may assume that u_n is a convergent sequence in $L^2(0, T; H_0^1)$ and weakly convergent in $L^2(0, T; H^2)$ with $\partial_t u_n$ weak-$*$ convergent in $L^2(0, T; L^2)$. The limit $u \in C([0, T); H_0^1) \cap L^2(0, T; H^2)$ is a weak solution of the Burgers equations on $[0, T)$. Note that continuity into H_0^1 follows from Lemma 1.12. □

Using the following lemma we can also conclude that the solution constructed above is smooth after the initial time. The proof applies equally in two or three dimensions, and is based on the fact that H^s is a Banach algebra for $s > d/2$. This is a standard type of result for parabolic systems, including the Navier–Stokes equations, (see for example Constantin and Foias (1988)), but our proof is slightly non-standard. Indeed we obtain explicit estimates on H^{s+1} norm of the Galerkin approximations away from the initial time rather than using boundedness of the solution in $L^2(0, T; H^{s+1})$ to deduce $u(\varepsilon) \in H^{s+1}$ for an arbitrarily small ε.

Lemma 4.5. *If the Galerkin approximations u_n are uniformly bounded in $L^\infty(\varepsilon, T; H^s) \cap L^2(\varepsilon, T; H^{s+1})$ for $s > 1/2$ and $\varepsilon \geq 0$, then they are also bounded uniformly in $L^\infty(\varepsilon', T; H^{s+1}) \cap L^2(\varepsilon', T; H^{s+2})$ for any $\varepsilon' \in (\varepsilon, T)$.*

Proof. It suffices to consider the case $\varepsilon = 0$. Taking the H^{s+1} product of (4.6) with $2tu_n$ and making use of the algebra property, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(t\|u_n\|_{H^{s+1}}^2) - \|u_n\|_{H^{s+1}}^2 + 2t\|\nabla u\|_{H^{s+1}}^2 \leq 2ct\|u_n\|_{H^{s+1}}^2\|\nabla u_n\|_{H^{s+1}}.$$

Hence, by Young's inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(t\|u_n\|_{H^{s+1}}^2) \leq \|u_n\|_{H^{s+1}}^2 + c^2t\|u_n\|_{H^{s+1}}^4.$$

This is a differential inequality of the form

$$\frac{\mathrm{d}X}{\mathrm{d}t} \leq f(t)(1 + c^2X)$$

from which we deduce that

$$\|u_n(t)\|_{H^{s+1}}^2 \leq \frac{1}{t}\int_0^t \|u_n(r)\|_{H^{s+1}}^2 \mathrm{e}^{c^2\int_r^t \|u_n\|_{H^{s+1}}^2} \,\mathrm{d}r.$$

Using the uniform bounds on $u_n \in L^2(0, T; H^{s+1})$, we see that for any $\varepsilon' \in (0, T)$ there is a uniform bound on $u_n \in L^\infty(\varepsilon', T; H^{s+1})$. In particular, we may assume that $\|u_n(\varepsilon')\|_{H^{s+1}}$ is uniformly bounded, since the approximants are continuous $u_n \in C([0, T); H^{s+1})$.

Taking the H^{s+1} product of (4.6) with $2u_n$ and integrating over (ε', T) yields

$$\int_{\varepsilon'}^t \|\nabla u(r)\|_{H^{s+1}}^2 \mathrm{d}r \leq \|u_n(\varepsilon')\|_{H^{s+1}}^2 + c\int_{\varepsilon'}^t \|u_n(r)\|_{H^{s+1}}^4 \mathrm{d}r,$$

so u_n is also uniformly bounded in $L^2(\varepsilon', T; H^{s+2})$ as required. □

Given uniform bounds on $u_n \in L^\infty(\varepsilon, T; H^{s+1}) \cap L^2(\varepsilon, T; H^{s+2})$ one can easily deduce uniform bounds on $\partial_t u_n \in L^2(\varepsilon, T; H^s)$, hence by the Aubin-Lions lemma, this regularity passes to the limit, u on (ε, T). A similar argument works for higher time derivatives (if we apply Lemma 4.5 repeatedly we can control sufficiently many spatial derivatives to obtain uniform bounds on a prescribed number of time derivatives). See the proof of Lemma 6.12 for details of an analogous argument. It follows that the solutions we constructed in Lemma 4.4 on a time interval $[0,T)$ are classical solutions on (ε,T) for any $\varepsilon>0$.

4.3.2 Global existence

We can now extend our H^1 solutions to all positive times using the maximum principle.

Corollary 4.6. *Let $\Omega\subset\mathbb{R}^d$ be a smooth bounded domain with $d=2$ or 3. Given $u_0\in H_0^1(\Omega)$ there exists a unique weak solution of the Burgers equations $u\in C([0,\infty);H_0^1)\cap L^2_{\mathrm{loc}}(0,\infty;H^2)$. Moreover u is a classical solution on $(0,\infty)\times\Omega$.*

Proof. Let $u\in C([0,T);H_0^1)\cap L^2(0,T;H^2)$ be a local solution from Lemma 4.4. Since u is a classical solution on $(0,T)$, for any fixed $s\in(0,T)$ it follows from Lemma 4.1 that

$$\sup_{t\in[s,T)}\|u(t)\|_{L^\infty}\le\|u(s)\|_{L^\infty}.$$

This gives us an additional H^1 estimate, by integrating the equations against $2\Delta u$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2&\le 2|((u\cdot\nabla)u,\Delta u)_{L^2}|-2\|\Delta u\|_{L^2}^2\\&\le 2\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}-2\|\Delta u\|_{L^2}^2\le\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2.\end{aligned}\tag{4.8}$$

It follows that

$$\|\nabla u\|_{L^2}^2\le\|\nabla u(s)\|_{L^2}^2\mathrm{e}^{(t-s)\|u(s)\|_{L^\infty}^2}.$$

So we can extend u from $[0,T)$ to a solution on $[0,T')$ where

$$T'=T+\frac{c}{\limsup_{t\to T}\|\nabla u\|^2}\ge T+c\|\nabla u(s)\|_{L^2}^{-2}\mathrm{e}^{-(T-s)\|u(s)\|_{L^\infty}^2}.$$

In particular, for any $R>0$ there exists $\tau_R>0$, depending only on $u(s)$ and R, such that when $T<R$ we may take $T'\ge T+\tau_R$. It follows that the solution u can be continued on $(0,R)$ for any $R>0$, as required. □

4.4 Global existence in $H^1(\mathbb{R}^d)$

In this section we shall extend the global existence results in H_0^1 (Corollary 4.6) to the whole space $\mathbb{R}^d$. We will use a "nested domains" approach, which is a familiar method for the Navier–Stokes equations; see Robinson et al. (2016) or Heywood (1988) for example. This proof should also work in other unbounded domains with smooth boundary, but we will only treat the case of the whole space here.

The main estimates we need in this section consist of the domain-independent energy estimates that comprise the following lemma.

Lemma 4.7. *For $d = 2, 3$, there exists a non-negative function*

$$F : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$$

that is non-decreasing in both dimensions such that for any smooth bounded domain $\Omega \subset \mathbb{R}^d$ the solution of the Burgers equations u constructed in Corollary 4.6 satisfies

$$\|u\|_{L^\infty(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)} + \|\partial_t u\|_{L^2(0,T;L^2)} \leq F(T, \|u_0\|_{H^1}), \tag{4.9}$$

for any $T > 0$ and any $u_0 \in H_0^1(\Omega)$.

Proof. We begin with a stronger version of the local estimates on the Galerkin approximations in $L^\infty H^1 \cap L^2 H^2$ (see (4.7)). Integrating the Galerkin ODEs (4.6) against $u_n - \Delta u_n$ yields

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2_{H^1} &+ \|\nabla u_n\|^2_{L^2} + \|\Delta u_n\|^2_{L^2} \\
&\leq \|u\|_{L^6}\|\nabla u\|_{L^3}\|u\|_{H^2} \leq \|u_n\|^{3/2}_{H^1_0}\|u_n\|^{3/2}_{H^2} \\
&\leq \frac{1}{2}\|\Delta u_n\|^2_{L^2} + \frac{1}{2}\|u_n\|^2_{L^2} + c\|u_n\|^6_{H^1_0} + c\|u_n\|^4_{H^1_0} + \frac{1}{2}\|\nabla u_n\|^2_{L^2},
\end{aligned}$$

where the constants appearing can be chosen independent of the domain. For example when we have used a Sobolev embedding we can take the constant from the embedding on the whole space.

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2_{H^1_0} + \|\nabla u_n\|^2_{L^2} + \|\Delta u_n\|^2_{L^2} \leq c(1 + \|u_n\|^2_{H^1_0})^3.$$

Solving this in the familiar way, we deduce that for some $T' > 0$ (that is a decreasing function of $\|u_0\|_{H_0^1}$) u_n is bounded in $L^\infty(0, T'; H_0^1)$ and $L^2(0, T'; H^2)$ by a constant G that is a non-decreasing function of $\|u_0\|_{H_0^1}$ and $T' > 0$. These estimates pass to the limit, i.e. to the solution u, and it suffices to find estimates for larger times and for the time derivative.

For the large-time estimates we note that there exists a time $S \in (0, T')$ such that

$$\|u(S)\|_{H^2}^2 \le \frac{1}{T'} \int_0^{T'} \|u(s)\|_{H^2}^2 \, \mathrm{d}s \le \frac{1}{T'} G(T', \|u_0\|_{H^1}).$$

Since $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ (and we can choose a constant independent of Ω, using the embedding $H^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$), by the maximum principle we obtain

$$\|u\|_{L^\infty(S,\infty;L^\infty)}^2 \le \frac{c}{T'} G(T', \|u_0\|_{H^1}).$$

A simple estimate akin to (4.8) now yields, for all $t > S$,

$$\|u(t)\|_{H_0^1}^2 \le \|u(S)\|_{H_0^1}^2 \mathrm{e}^{\frac{c}{2} tG} \le G(T') \mathrm{e}^{\frac{ct}{2T'} G(T')}$$

and

$$\int_S^t \|u(s)\|_{H^2}^2 \, \mathrm{d}s \le \|u(S)\|_{H_0^1}^2 + (1 + \frac{c}{T'} G(T')) \int_S^t \|u(s)\|_{H_0^1}^2 \, \mathrm{d}s.$$

It remains to estimate the time derivative. This can be done in the same way as in the proof of Lemma 4.4. Indeed from the estimates above we have bounds on Δu and $(u \cdot \nabla) u$ in $L^2(0, T; L^2)$ independent of Ω. $\square$

By virtue of the estimates in Lemma 4.7 we can now apply a "nested domains" argument to construct solutions on the whole space.

Theorem 4.8. *If $u_0 \in H^1(\mathbb{R}^d)$, there exists a unique weak solution of the Burgers equations $u \in C_w([0, \infty); H^1(\mathbb{R}^d)) \cap L^2(0, \infty; H^2(\mathbb{R}^d))$.*

Proof. Fix $T > 0$ and let $\{B_n\}_{n=1}^\infty$ be a nested collection of open balls $B_n \subset B_{n+1}$ such that $\mathbb{R}^d = \bigcup_n B_n$. Let $(v_n)_{n=1}^\infty$ be a sequence of functions $v_n \in H_0^1(B_n)$ such that $Ev_n \to u_0$ in $H^1(\mathbb{R}^d)$, where $E = E_n$ denotes extension by zero from B_n to $\mathbb{R}^d$. We may also assume that $\|v_n\|_{H_0^1} \le 2\|u_0\|_{H^1}$. By Corollary 4.6 for each n there exist a weak solution u_n on $[0, \infty)$ in the domain B_n corresponding to the initial data v_n.

By Lemma 4.7 there is a non-negative function F such that for all n and all $T > 0$,

$$\|u_n\|_{L^\infty(0,T;H^1(B_n))} + \|u_n\|_{L^2(0,T;H^2(B_n))} + \|\partial_t u_n\|_{L^2(0,T;L^2(B_n))} \\ \leq F(T, \|v_n\|_{H_0^1}) \leq F(T, 2\|u_0\|_{H^1}).$$

It follows that for each n, the restrictions $(\chi_{B_n} u_m)_{m>n}$ form a bounded sequence in $L^2(0,T;H^2(B_n))$ with derivatives $\chi_{B_n}\partial_t u_m$ bounded independent of m in $L^2(0,T;L^2(B_n))$. Applying the Aubin-Lions lemma countably many times (once on each domain B_n) we may now construct a subsequence relabelled u_n such that $\chi_{B_n} u_m$ converges in $L^2(0,T;H^1(B_n))$ for every n to a limit $u \in L^\infty(0,T;H^1(\mathbb{R}^d)) \cap L^2(0,T;H^2(\mathbb{R}^d))$.

The limit u is a weak solution of the Burgers equations on each B_n. Since the definition of weak solution only uses compactly supported test functions this is sufficient to deduce that u is a local solution on the whole space $\mathbb{R}^d$.

The fact that this solution can be extended indefinitely follows from the fact that $T > 0$ can be chosen arbitrarily large, and from uniqueness (Lemmas 4.2 and 4.3). □

4.5 Solutions in $L^p(\Omega)$ and $L^p \cap L^2(\mathbb{R}^d)$ $p > d$

We can now extend the results above to construct global solutions for initial data in L^p, $p > d$, in bounded domains or $L^p \cap L^2(\mathbb{R}^d)$. In proving the case for bounded domains in the following theorem we will go out of our way to make "domain independent" estimates as we did in the H^1 case. The case of the whole space will then follow in an analogous way to the $H^1(\mathbb{R}^d)$ case above.

Our local existence argument is based on L^p estimates applicable to the Navier–Stokes equations for $p \geq 3$. See for example the expositions of local existence of weak solutions of that system by Robinson and Sadowski (2014) and Robinson et al. (2016). These are based on L^p estimates similar to those considered by Beirão da Veiga (1987).

At present we have not completed a study of the case $p = d$, this is perhaps a subject for future work. We will make a brief comment at the end of the section on why it seems reasonable to expect a similar local existence result (in bounded or unbounded domains) to hold when $p = d = 3$.

It is worth emphasising that in the analysis here, we rely on the extra decay condition $u_0 \in L^2$ in the case of an unbounded domain. For the Navier–Stokes equations, a more difficult approach gives local existence and uniqueness in for initial data in $L^p(\mathbb{R}^d)$, where $p > d$ i.e. without assuming finite kinetic energy. See, for example, Fabes et al. (1972). It would be interesting to investigate whether a similar result holds for the Burgers equations.

We will treat the cases of two and three-dimensional bounded domains separately, before sketching the extension to unbounded domains, which is very similar, given our results in $H^1(\mathbb{R}^d)$.

4.5.1 Bounded domains in three dimensions

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain and $u_0 \in L^p$ for some $p \in (3, \infty)$. There exists a corresponding unique weak solution $u \in C([0,\infty); L^2) \cap L^2_{\text{loc}}(0,\infty; H^1_0)$ and additionally,*

$$u \in L^\infty(0,T; L^p) \cap L^p(0,T; L^{3p}),$$

for all $T > 0$.

Proof. Fix $p > 3$ and $u_0 \in L^p(\Omega)$. Let $(v_n)_{n=1}^\infty \subset C_c^\infty(\Omega)$ be a sequence of functions converging to u_0 in L^p. By Corollary 4.6, each v_n gives rise to a unique global solution $u_n \in C([0,\infty); H^1_0) \cap L^2_{\text{loc}}(0,\infty; H^2_0)$, moreover each u_n is a classical solution. Uniqueness will follow from Lemmas 4.2 , so it suffices to show that a subsequence of (u_n) converges to a solution.

The key step is to prove that (u_n) is uniformly bounded in $L^\infty(0,T; L^p)$ for any $T > 0$. In this calculation we will use the following simple observations. If $f \in C^2(\Omega)$ then

$$\begin{aligned}\nabla f \cdot \nabla(f|f|^{p-2}) &= |\nabla f|^2|f|^{p-2} + (p-2)\sum_{1\le k\le d}|f_k \nabla f_k|^2|f|^{p-4}\\ &= |\nabla f|^2|f|^{p-2} + \frac{p-2}{4}|\nabla|f|^2|^2|f|^{p-4}, \quad (4.10)\end{aligned}$$

and if $f \in W_0^{1,p}$ then, using the embedding $H^1 \hookrightarrow L^6$,

$$\begin{aligned}\|f\|_{L^{3p}} \le c\||f|^{p/2}\|_{L^6}^{2/p} &\le c\|f\|_{L^p} + c\|\nabla |f|^{p/2}\|_{L^2}^{2/p} \\ &\le c\|f\|_{L^p} + c\left(\int_\Omega |\nabla f|^2 |f|^{p-2}\right)^{1/p}. \qquad (4.11)\end{aligned}$$

Note that we have added an extra term on the right-hand side in order to avoid using a Poincaré inequality. As a result we may take the constant c independent of the domain Ω by using an embedding constant for the whole space. This is not important here but will be very useful when generalising to unbounded domains.

With a little more difficulty it can be shown that there exists $c > 0$ independent of Ω such that

$$\|f\|_{L^{3p}} \le c\left(\int_\Omega |\nabla f|^2 |f|^{p-2}\right)^{1/p}. \qquad (4.12)$$

See for example the discussion of the Navier–Stokes equations in $L^3(\mathbb{R}^3)$ Chapter 11 of Robinson et al. (2016). However, in the case of the Burgers equations in $L^p(\mathbb{R}^3)$ for $p > 3$, (4.11) is sufficient for our purposes.

Integrating the equations satisfied by u_n against $u_n|u_n|^{p-2}$ and applying these observations now yields the estimates

$$\begin{aligned}\frac{1}{p}\int_\Omega |u_n(t)|^p + \int_0^t \int_\Omega |\nabla u_n|^2 |u_n|^{p-2} + \frac{p-2}{4}\int_0^t\int_\Omega |\nabla |u_n|^2|^2 |u_n|^{p-4} \\ \le \frac{1}{p}\int_\Omega |v_n|^p + \int_0^t \int_\Omega |u_n|^p |\nabla u_n| \\ \le \frac{1}{p}\int_\Omega |v_n|^p + \frac{1}{2}\int_0^t\int_\Omega |\nabla u_n|^2 |u_n|^{p-2} + \frac{1}{2}\int_0^t\int_\Omega |u_n|^{2+p}. \qquad (4.13)\end{aligned}$$

Since $\|u_n\|_{L^{p+2}} \le \|u_n\|_{L^p}^{(p-1)/(p+2)} \|u_n\|_{L^{3p}}^{3/(p+2)}$ it follows from (4.11) that

$$\begin{aligned}\int_\Omega |u_n|^{2+p} &\le c\|u_n\|_{L^p}^{p-1}\left(\int_\Omega |\nabla u_n|^2 |u_n|^{p-2}\right)^{3/p} + c\|u_n\|_{L^p}^{p+2} \\ &\le c\|u_n\|_{L^p}^{p(p-1)/(p-3)} + c\|u_n\|_{L^p}^{p+2} + \frac{1}{2}\left(\int_\Omega |\nabla u_n|^2 |u_n|^{p-2}\right) \\ &\le c(\|u_n\|_{L^p}^p + 1)^{(p-1)/(p-3)} + \frac{1}{2}\left(\int_\Omega |\nabla u_n|^2 |u_n|^{p-2}\right),\end{aligned}$$

where we have used the fact that $(p-1)/(p-3) > (p+2)/p$. Hence from (4.13) we deduce that

$$\frac{1}{p}\int_\Omega |u_n(t)|^p + \frac{1}{4}\int_0^t \int_\Omega |\nabla u_n|^2 |u_n|^{p-2} \\ \leq \frac{1}{p}\int_\Omega |v_n|^p + c\int_0^t (\|u_n(s)\|_{L^p}^p + 1)^{(p-1)/(p-3)}\,\mathrm{d}s. \quad (4.14)$$

Solving this integral inequality yields

$$\|u_n(t)\|_{L^p}^p \leq \frac{\|v_n\|_{L^p}^p + 1}{\left(1 - \frac{2c}{p-3}(\|v_n\|_{L^p}^p + 1)^{2/(p-3)} t\right)^{(p-3)/2}} - 1. \quad (4.15)$$

It now follows from (4.11) and (4.14) that for any T satisfying

$$0 < T < \frac{p-3}{2c}(\|v_n\|_{L^p}^p + 1)^{-2/(p-3)},$$

(u_n) is uniformly bounded in $L^\infty(0,T;L^p) \cap L^p(0,T;L^{3p})$.

These estimates are sufficient to deduce that u_n is bounded independent of n, in $L^2(0,T;H_0^1)$. Indeed, integrating the Burgers equations against u_n, we obtain

$$\begin{aligned}\frac{1}{2}\|u_n(t)\|_{L^2}^2 - \frac{1}{2}\|v_n\|_{L^2}^2 + \int_0^t \|\nabla u_n(s)\|_{L^2}^2\,\mathrm{d}s &\leq \int_0^t \int_\Omega |(u_n \cdot \nabla)u_n \cdot u_n| \\ &\leq \int_0^t \|u_n\|_{L^p}\|\nabla u_n\|_{L^2}\|u_n\|_{L^{2p/(p-2)}} \\ &\leq \int_0^t \|\nabla u_n(s)\|_{L^2}\|u_n\|_{L^p}\|u_n\|_{L^2}^{(p-3)/p}\|u_n\|_{L^6}^{3/p} \\ &\leq \frac{1}{2}\int_0^t \|\nabla u_n\|_{L^2}^2 + c\int_0^t (\|u_n\|_{L^p}^{2p/(p-3)} + 1)\|u_n\|_{L^2}^2 \quad (4.16)\end{aligned}$$

so $u_n \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1)$ is bounded independent of n. Now integrating the equations against $v \in H_0^1$, using the fact that u_n is uniformly bounded in $L^\infty(0,T;L^3)$ (since $u_n \in L^\infty(0,T;L^p) \cap L^\infty(0,T;L^2)$) we obtain:

$$|(\partial_t u_n, v)| \leq |\langle \Delta u_n, v\rangle_{H^{-1}\times H_0^1}| + \|u_n\|_{L^3}\|\nabla u_n\|_{L^2}\|v\|_{H_0^1} \in L^2(0,T), \quad (4.17)$$

so $\partial_t u_n$ is uniformly bounded in $L^2(0,T;H^{-1})$.

We can now apply the Aubin-Lions lemma again, and pass to a subsequence that converges to a weak solution u of the Burgers equations such that $u \in L^\infty(0,T;L^p)$.

By the results in the previous section this solution u can be extended to a classical solution on $(0,\infty)$ and $u \in L^\infty(s,\infty;L^\infty)$ for any $s > 0$. Since Ω is a bounded domain, it follows that $u \in L^\infty(0,\infty;L^p)$ hence, by Lemma 4.2, u is the unique weak solution with this regularity. □

4.5.2 Bounded domains in two dimensions

The proof of the following theorem is similar to the proof of Theorem 4.9, above, so we will only sketch the proof.

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $u_0 \in L^p$ for some $p \in (2,\infty)$. There exists a corresponding unique weak solution $u \in C([0,\infty);L^2) \cap L^2_{\mathrm{loc}}(0,\infty;H^1_0)$ with the additional regularity*

$$u \in L^\infty(0,T;L^p) \cap L^p(0,T;L^r)$$

for all $T > 0$ and any $r \in (p^2/(p-2),\infty)$.

Proof. As before, let (v_n) be a sequence of C_c^∞ functions such that $v_n \to u_0$ in L^p and $n \to \infty$. Let u_n be the global solution corresponding to v_n, constructed in Corollary 4.6. Proceeding as in (4.13) we obtain

$$\frac{1}{p}\int_\Omega |u_n(t)|^p + \frac{1}{2}\int_0^t\int_\Omega |\nabla u_n|^2|u_n|^{p-2} \le \frac{1}{p}\int_\Omega |v_n|^p + \frac{1}{2}\int_0^t\int_\Omega |u_n|^{2+p}. \tag{4.18}$$

Let r satisfy the hypothesis in the statement and let $\sigma := 2r/p \in (2,\infty)$ (to save notation) then

$$\frac{2\sigma}{p(\sigma-2)} < 1, \tag{4.19}$$

and $\sigma p/2 > \sigma + p > 2 + p$.

To estimate $\|u_n\|_{p+2}^{p+2}$, we interpolate $\|u_n\|_{p+2}$ between the L^p and L^r ($L^{\sigma p/2}$) norms, to obtain

$$\int_\Omega |u_n|^{p+2} \le \|u_n\|_{L^p}^{\frac{\sigma p-2p-4}{\sigma-2}} \left(\int_\Omega |u_n|^{\sigma p/2}\right)^{\frac{4}{p\sigma-2p}}.$$

By an observation similar to (4.11), using the embedding $H_0^1 \hookrightarrow L^r$ there exists $c > 0$ such that

$$\left(\int_\Omega |u_n|^{\sigma p/2}\right)^{1/\sigma} \leq c\|u_n\|_{L^p}^{p/2} + c\left(\int_\Omega |\nabla u_n|^2 |u_n|^{p-2}\right)^{1/2}. \tag{4.20}$$

It follows that

$$\int_\Omega |u_n|^{p+2} \leq c\|u_n\|_{L^p}^{\frac{p(\sigma p-2p-4)}{\sigma p-2p-2\sigma}} + c\|u_n\|_{L^p}^p + \frac{1}{2}\int_\Omega |\nabla u_n|^2 |u_n|^{p-2},$$

note that this step depends on (4.19), or (equivalently) the lower bound on r in the statement. Using this in (4.18), we obtain

$$\frac{1}{p}\|u_n(t)\|_{L^p}^p + \frac{1}{4}\int_0^t \int_\Omega |\nabla u_n|^2 |u_n|^{p-2} \leq \frac{1}{p}\|v_n\|_{L^p}^p + c\int_0^t (\|u_n\|_{L^p}^p + 1)^\alpha \tag{4.21}$$

where $\alpha = (\sigma p - 2p - 4)/(\sigma p - 2p - 2\sigma) > 1$. Therefore for some $T > 0$ $u_n \in L^\infty(0,T;L^p)$ and $u_n \in L^p(0,T;L^{\sigma p/2}) = L^p(0,T;L^r)$ with respective estimates independent of n.

In order to prove existence in 2D we can use these L^p bounds to find L^2 estimates, as in the 3D case. The argument proceeds as before, except that instead of interpolating $\|u\|_{L^{2p/(p-2)}}$ between L^2 and L^6, we use L^2 and L^σ respectively, where σ satisfies (4.19) (see the estimate of the nonlinear term in Lemma 4.3).

Global existence in 2D is analogous to the $3D$ case, and uniqueness follows from Lemma 4.3. $\square$

4.5.3 Extending Theorems 4.9 and 4.10 to the whole space

The proofs of Theorems 4.9 and 4.10 can be adapted to the case $\Omega = \mathbb{R}^d$. Indeed, given $u_0 \in L^p(\mathbb{R}^d)$ we can consider a sequence of approximations $v_n \in C_c^\infty(\mathbb{R}^d)$ that converge to u_0 in L^p. The corresponding solutions u_n from Theorem 4.8 can be estimated in L^p as in the previous sections. That is, if $d = 3$, there exist uniform bounds on u_n in

$$L^\infty(0,T;L^p(\mathbb{R}^3)) \cap L^p(0,T;L^{3p}(\mathbb{R}^3))$$

for some $T > 0$, and if $d = 2$, u_n is uniformly bounded in

$$L^\infty(0,T;L^p(\mathbb{R}^2)) \cap L^p(0,T;L^r(\mathbb{R}^2))$$

for any $r > p^2/(p-2)$.

If additionally $u_0 \in L^2(\mathbb{R}^d)$, we can also obtain uniform bounds on u_n in $L^\infty(0,T;L^2(\mathbb{R}^d))$ and $L^2(0,T;H^1(\mathbb{R}^d))$. It follows that $\partial_t u_n$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$ for any domain $\Omega \subseteq \mathbb{R}^d$. We then obtain a subsequence, relabelled u_n, converging to a solution on each of a nested sequence of bounded domains, by the Aubin-Lions lemma.

The limit u is a weak solution in $L^\infty(0,T;L^2(\mathbb{R}^d))$ and $L^2(0,T;H^1(\mathbb{R}^d))$. We can also obtain bounds in L^p by arguing as in the following paragraph.

The solution u is in $H^1(\mathbb{R}^d)$ immediately after the initial time and so can be uniquely extended to a global solution, by Theorem 4.8. Moreover for any $\varepsilon' > 0$, $u \in L^2_{\text{loc}}(\varepsilon',\infty;H^2(\mathbb{R}^d))$, hence for $d = 2,\ 3$, there exists $\varepsilon > 0$ such that $u(\varepsilon) \in L^\infty(\mathbb{R}^d)$. It follows that $u \in L^\infty(\varepsilon,\infty;L^\infty)$ by the maximum principle. Following the estimates (4.13) and (4.18), we obtain

$$\begin{aligned}\|u(t)\|_{L^p}^p + \frac{p}{2}\int_\varepsilon^t \int_{\mathbb{R}^d} |\nabla u|^2 |u|^{p-2} \\ \le \|u(\varepsilon)\|_{L^p}^p + \frac{p}{2}\|u\|^2_{L^\infty(\varepsilon,\infty;L^\infty)} \int_\varepsilon^t \|u(s)\|_{L^p}^p \,\mathrm{d}s,\end{aligned}$$

so $u \in L^\infty(0,\infty;L^p)$ and for any $T > 0$ $u \in L^p(0,T;L^{3p})$ or $u \in L^p(0,T;L^r)$ for any $r > p^2/(p-2)$ in dimensions three and two respectively.

Based on this sketch, the following theorem can be proved.

Theorem 4.11. *For $d = 2$ or 3 let and $u_0 \in L^p \cap L^2(\mathbb{R}^d)$ for some $p > d$ then there exists a unique weak solution $u \in C([0,\infty);L^2) \cap L^2_{\text{loc}}(0,\infty;H^1)$ with the additional regularity that for all $T > 0$*

$$u \in L^\infty(0,T;L^p) \cap L^p(0,T;L^{3p})$$

if $d = 3$, or

$$u \in L^\infty(0,T;L^p) \cap L^p(0,T;L^r)$$

for any $r > p^2/(p-2)$ if $d = 2$.

The compactness arguments above relied on the uniform estimates in L^2 based on the assumption that $u_0 \in L^2(\mathbb{R}^d)$. It is not immediately clear whether this hypothesis can easily be dropped. However it is reasonable to expect that a more involved argument might exist to overcome this difficulty (following the treatment of the Navier–Stokes equations in $L^p(\mathbb{R}^n)$ by Fabes et al. (1972), for example).

4.5.4 Remarks on the case $p = 3$ in three dimensions

We now briefly discuss the possibility of extending Theorem 4.11 to the critical case of L^3 in three dimensions.

Consider, for the sake of simplicity, the case of a smooth bounded domain $\Omega \subset \mathbb{R}^3$ and initial data $u_0 \in L^3(\Omega)$. Following the proof of Theorem 4.9, let $(v_n)_{n=1}^{\infty} \subset C_c^{\infty}(\Omega)$ be a sequence of smooth approximations converging to u_0 in L^p and the corresponding solutions $u_n \in L^\infty(0,\infty;H^1) \cap L^2(0,\infty;H^2)$ of the Burgers equations from Corollary 4.6.

It seems reasonable that to expect that the short-time L^3 estimates proved using a data splitting argument for the Navier–Stokes equations (see Chapter 11 of Robinson et al. (2016)) can be adapted to the Burgers equations. The next chapter contains an example of such an argument in $H^{1/2}$. Assuming that this is the case, our approximations u_n can be bounded uniformly in $L^\infty(0,T;L^3)$ and $L^3(0,T;L^9)$ for some $T > 0$ depending on $\|u_0\|_{L^3}$.

In order to construct a solution using these estimates we have the following energy calculation

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|_{L^2}^2 + \|\nabla u_n\|_{L^2}^2 &\le |((u_n\cdot\nabla)u_n, u_n)| \le \|\nabla u_n\|_{L^2}\|u_n\|_{L^9}\|u_n\|_{L^{18/7}} \\ &\le c\|\nabla u_n\|_{L^2}^{4/3}\|u_n\|_{L^9}\|u_n\|_{L^2}^{2/3} \le \frac{1}{2}\|\nabla u_n\|_{L^2}^2 + c\|u_n\|_{L^9}^3\|u_n\|_{L^2}^2. \quad (4.22)\end{aligned}$$

Hence u_n is also uniformly bounded in $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ and, after passing to a subsequence, we may assume that u_n converges to a weak solution in $L^2(0,T;L^2)$.

To prove uniqueness in the case of the Navier–Stokes equations in L^3 it suffices to show that a Leray–Hopf weak solution corresponding to $u_0 \in L^3$ satisfies a Serrin condition, that is

$$u \in L^r(0,T;L^s)$$

for $r \geq 2$ and $s > 3$ such that

$$\frac{2}{r} + \frac{3}{s} = 1.$$

The relevant choice of exponents here is $r = 3,\ s = 9$. A weak solution of the Navier–Stokes equations satisfying such a condition becomes smooth immediately after the initial time, and is unique. See for example Chapter 8 of Robinson et al. (2016), Galdi (2000) or Serrin (1963).

If u_0 is regular enough that the Burgers equations admit a corresponding weak solution satisfying a Serrin-type condition then it is reasonable to expect that a similar uniqueness result holds. For example we have the following a priori uniqueness estimates. Given $u, v \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ solutions of the Burgers equations such that $u, v \in L^3(0,T;L^9)$, then (following (4.22)) $w = u - v$ satisfies

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 &\leq |((u \cdot \nabla)w, w)_{L^2}| + |((w \cdot \nabla)w, v)_{L^2}| \\ &\quad + |((\nabla \cdot w)v, w)_{L^2}| \\ &\leq \frac{1}{2}\|\nabla w\|_{L^2}^2 + c(\|v\|_{L^9}^3 + \|u\|_{L^9}^3)\|w\|_{L^2}^2,\end{aligned}$$

from which uniqueness formally follows, in the case $w(0) = 0$.

Chapter 5

Well-posedness for the Burgers equations in $H^{1/2}(\mathbb{T}^3)$

5.1 Introduction

In this chapter we will discuss the global well posedness of the Burgers equations in the space $H^{1/2}(\mathbb{T}^3)$, which is a critical space in three-dimensions, in the sense that on $\mathbb{R}^3$ the norm $H^{1/2}$ is invariant under the natural scaling of the Burgers equations. Notice that since $H^{1/2} \hookrightarrow L^3$ is a critical Sobolev embedding (i.e. $H^{1/2}$ is not embedded in L^p for $p > 3$) these results are a genuine addition to the work in the previous chapter.

This chapter is based on a paper by Pooley and Robinson (2016a). Here we revisit the proof therein, that the Burgers equations are globally well-posed in $H^{1/2}(\mathbb{T}^3)$. Following that paper, we work in the setting of periodic boundary conditions, which we have not illustrated in the previous chapter. In fact there are some interesting peculiarities to this case, as a consequence of the Burgers evolution not conserving momentum; a fact that in other domains can be dealt with by applying a suitable boundary condition.

We will use an analogous notion of weak solution as we did in the cases of a bounded domain and the whole space (4.5), noting however that the compact spatial support hypothesis on the test functions is redundant.

We call a weak solution with the additional regularity

$$u \in C([0,T]; H^{1/2}) \cap L^2(0,T; H^{3/2})$$

and $\partial_t u \in L^2(0,T; H^{-1/2})$ an $H^{1/2}$ solution of the Burgers equations. As usual, we call such a solution "global" if it can be continued on $[0,T]$ for any $T > 0$. The main result of this chapter is the following theorem.

Theorem 5.1. *Given $u_0 \in H^{1/2}(\mathbb{T}^3)$, there exists a unique global $H^{1/2}$ solution of the Burgers equations corresponding to the initial data u_0. Moreover, u is a classical solution for $t > 0$ and*

$$u \in C^1((0,\infty); C(\mathbb{T}^3)) \cap C((0,\infty); C^2(\mathbb{T}^3)).$$

Global existence and smoothness follows, as in the last chapter, from global existence in H^1 (a consequence of the maximum principle). To prove local well-posedness in $H^{1/2}$, we split the equations into a linear (heat) part with initial data and a nonlinear part with vanishing data. This is in keeping with well known results for the Navier–Stokes equations in critical spaces, see for example Marín-Rubio et al. (2013), Chemin, Desjardins, Gallagher, and Grenier (2006), Calderón (1990) and Fabes et al. (1972).

As mentioned above, our estimates in the periodic case in $H^{1/2}$ are sometimes complicated by the fact that the Burgers equations do not conserve momentum i.e.

$$P(t) = \int_{\mathbb{T}^3} u(x,t)\,\mathrm{d}x$$

(or equivalently the zeroth Fourier coefficient). In the case of the Navier–Stokes equations, which do conserve momentum, it is usual to consider solutions with zero average to simplify the relevant estimates, but we must be more careful here.

For data in $H^1(\mathbb{T}^3)$ we can proceed as we did for unbounded domains, in the previous chapter. In other words, the failure of momentum conservation does not affect the proof based on using $u - \Delta u$ as a "test function" (in the sense of a priori estimates). Note that in Pooley and Robinson (2016a), our estimates in H^1 were more similar to the ones below for the case of solutions in $H^{1/2}$. For data in $H^{1/2}$, on the other hand, it will be simplest to follow our approach from Pooley and Robinson (2016a), where we estimate the growth of the momentum using Lemma 5.5 below.

5.2 Proof of Theorem 5.1

To prove Theorem 5.1 it will be enough to prove a suitable local existence and uniqueness result, given that global existence and smoothness then follow from results analogous to those in Section 4.3. Indeed, the following lemma can be proved using analogous methods to the ones in Section 4.3, except that for local existence we must use estimates similar to the "domain independent" ones from Lemma 4.7. Alternatively, we can use the same momentum estimates as we shortly will in the $H^{1/2}$ case (see Pooley and Robinson (2016a) for details).

Lemma 5.2. *Given $u_0 \in H^1(\mathbb{T}^3)$, there exists a unique global weak solution $u \in C([0,\infty); H^1) \cap L^2(0,\infty; H^2)$. Moreover, except at the initial time, $u \in C^1((0,\infty); C^0) \cap C((0,\infty); C^2)$ is a classical solution.*

Therefore, to prove Theorem 5.1, it suffices to prove the following lemmas.

Lemma 5.3. *For $u_0 \in H^{1/2}$ there exists $T > 0$ and an $H^{1/2}$ solution*

$$u \in C([0,T]; H^{1/2}) \cap L^2(0,T; H^{3/2})$$

to the Burgers equations, corresponding to the initial data u_0.

Lemma 5.4. *If u, v are both $H^{1/2}$ solutions to the Burgers equations on $[0,T) \times \mathbb{T}^3$ corresponding to the same initial data $u_0 \in H^{1/2}$ then $u = v$.*

5.2.1 Proof of Lemmas 5.3 and 5.4

As in the case of a bounded domain, we will construct a solution using Galerkin approximations, this time based on Fourier truncations. For $n > 1$ let $u_n = P_n u_n$ be the solution of the following system of ODEs:

$$\frac{\partial u_n}{\partial t} + P_n[(u_n \cdot \nabla)u_n] - \Delta u_n = 0, \tag{5.1}$$

with

$$u_n(0) = P_n u_0. \tag{5.2}$$

A classical solution exists on a time interval $[0, T_n)$ for some $T_n > 0$. Without loss of generality we will take T_n to be the maximal existence time, i.e. the solution u_n cannot be extended uniquely to a solution beyond that time. It follows that either $T_n = \infty$ or $\|u_n(t)\|_{L^2} \to \infty$ as $t \uparrow T_n$.

As discussed above, the non-conservation of momentum in solutions to the Burgers equations is a significant technical issue in the analysis for the periodic case. The following lemma and remarks give us sufficient control of the momentum of the solutions to (5.1) to overcome these difficulties.

Lemma 5.5. *Let u, v be solutions of (5.1), with initial data u_0 and v_0 respectively. If $w = u - v$ and $w_0 = u_0 - v_0$ then*

$$\left| \int_{\mathbb{T}^3} w(x,t) - w_0(x) \,\mathrm{d}x \right| \le 8\pi^3 \int_0^t \|w(s)\|_{1/2} (\|u(s)\|_{1/2} + \|v(s)\|_{1/2}) \,\mathrm{d}s. \tag{5.3}$$

In particular, taking $v \equiv 0$ yields

$$\left| \int_{\mathbb{T}^3} u(x,t) \,\mathrm{d}x \right| \le 8\pi^3 \int_0^t \|u(s)\|_{1/2}^2 \,\mathrm{d}s + \left| \int_{\mathbb{T}^3} u_0(x) \,\mathrm{d}x \right|. \tag{5.4}$$

Proof. For $k \in \mathbb{Z}^3$ denote the kth Fourier coefficients of u, v and w by $\hat{u}_k$, $\hat{v}_k$ and $\hat{w}_k$ respectively. Considering the form of the equations satisfied by u and v, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} w(x,t) \,\mathrm{d}x &= -\int_{\mathbb{T}^3} (u \cdot \nabla) w + (w \cdot \nabla) v \,\mathrm{d}x \\ &= -8\pi^3 \mathrm{i} \sum_{k \in \mathbb{Z}^3} \left(\overline{\hat{u}_k(t)} \cdot k \right) \hat{w}_k(t) + \left(\overline{\hat{w}_k(t)} \cdot k \right) \hat{v}_k(t), \end{aligned}$$

Hence

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} w(x,t) \,\mathrm{d}x \right| &\le 8\pi^3 \sum_{k \in \mathbb{Z}^3} |\hat{w}_k||k|(|\hat{u}_k| + |\hat{v}_k|) \\ &\le 8\pi^3 \|w(t)\|_{1/2} (\|u(t)\|_{1/2} + \|v(t)\|_{1/2}), \end{aligned}$$

then (5.3) follows after integrating with respect to t. □

We will use this lemma to control the failure of equivalence of the norm $\|\cdot\|_{H^s}$ and the seminorm $\|\cdot\|_s$ for solutions of (5.1) as follows:

$$\|u_n(t)\|_s \le \|u_n(t)\|_{H^s} \le c\|u_n(t)\|_s + c \int_0^t \|u_n\|_{1/2}^2 \,\mathrm{d}s + c\|u_0\|_{L^1}, \tag{5.5}$$

for some $c > 0$ depending only on s. Here we have used the fact that $\int_{\mathbb{T}^3} P_n u_0 = \int_{\mathbb{T}^3} u_0$. Note that we will occasionally use the equivalence of the seminorms $\|\cdot\|_s$ and $\|\cdot\|_{\dot{H}^s}$ when applicable. In particular for estimating the derivatives of sufficiently regular functions e.g. $\|\nabla u_n\|_{L^6} \leq c\|u_n\|_2$.

Proof (of Lemma 5.3). In order to derive the $H^{1/2}$ estimates necessary to pass to a convergent subsequence of (u_n), we follow Marín-Rubio et al. (2013) (see also Chemin et al. (2006), Calderón (1990) and Fabes et al. (1972)) and split (5.1) into a heat part, and a nonlinear part with zero initial data. Let v be the periodic solution of the heat equation with initial data u_0, then $v_n \coloneqq P_n v$ satisfies

$$\frac{\partial}{\partial t} v_n + \Delta v_n = 0, \ v_n(0) = P_n u_0.$$

Let $w_n \coloneqq u_n - v_n$, then w_n satisfies

$$\frac{\partial}{\partial t} w_n + P_n[(u_n \cdot \nabla) u_n] - \Delta w_n = 0, \ w_n(0) = 0. \tag{5.6}$$

For v_n and $t \in [0, T_n)$ we have the estimate

$$\sup_{s \in [0,t]} \|v_n(s)\|_{H^{1/2}}^2 + 2 \int_0^t \|v_n(s)\|_{H^{3/2}}^2 \, \mathrm{d}s \leq \|P_n u_0\|_{H^{1/2}}^2. \tag{5.7}$$

Integrating (5.6) against $\Lambda^1 w_n$, gives

$$\begin{aligned} &\|w_n(t)\|_{1/2}^2 + 2 \int_0^t \|w_n(s)\|_{3/2}^2 \, \mathrm{d}s \\ &\leq \int_0^t \|u_n(s)\|_{L^6} \|\nabla u_n(s)\|_{L^2} \|\Lambda^1 w_n(s)\|_{L^3} \, \mathrm{d}s \\ &\leq c_1 \int_0^t \|u_n(s)\|_{H^1} \|u_n(s)\|_1 \|w_n(s)\|_{3/2} \, \mathrm{d}s =: I_0. \end{aligned} \tag{5.8}$$

For some $c_1 > 0$. Now by Lemma 5.5 and the definition of w_n,

$$\begin{aligned} \|u_n(t)\|_{H^1} \|u_n(t)\|_1 &\leq c_2 \left(\|u_n(t)\|_1 + \int_0^t \|u_n(s)\|_{1/2}^2 \, \mathrm{d}s + \|u_0\|_{L^1} \right) \|u_n(t)\|_1 \\ &\leq 2c_2 (\|v_n(t)\|_1^2 + \|w_n(t)\|_1^2) \\ &\quad + c_2 (\|v_n(t)\|_1 + \|w_n(t)\|_1) \left(\int_0^t \|u_n(s)\|_{1/2}^2 \, \mathrm{d}s + \|u_0\|_{L^1} \right) =: I_1 + I_2 \end{aligned}$$

for some $c_2 > 0$. To estimate $I_1 \times \|w_n\|_{3/2}$ we apply Young's inequality,

$$\begin{aligned}&\|w_n(t)\|_{3/2}(\|v_n(t)\|_1^2 + \|w_n(t)\|_1^2)\\&\quad\leq \frac{1}{4c_1c_2}\|w_n(t)\|_{3/2}^2 + c\|v_n(t)\|_1^4 + \|w_n(t)\|_{3/2}\|w_n(t)\|_1^2,\end{aligned}$$

for some $c > 0$. Also by several applications of Young's inequality, we estimate $I_2 \times \|w_n\|_{3/2}$ as follows:

$$\begin{aligned}&\|w_n(t)\|_{3/2}(\|v_n(t)\|_1 + \|w_n(t)\|_1)\left(\int_0^t \|u_n(s)\|_{1/2}^2\,\mathrm{d}s + \|u_0\|_{L^1}\right)\\&\leq \frac{1}{2}\|w_n(t)\|_{3/2}\left(\|v_n(t)\|_1^2 + \|w_n(t)\|_1^2\right)\\&\quad + \|w_n(t)\|_{3/2}\left(\int_0^t \|u_n(s)\|_{1/2}^2\,\mathrm{d}s + \|u_0\|_{L^1}\right)^2\\&\leq \frac{1}{2c_1c_2}\|w_n(t)\|_{3/2}^2 + c\|v_n(t)\|_1^4 + \frac{1}{2}\|w_n(t)\|_{3/2}\|w_n(t)\|_1^2\\&\quad + c\left(\int_0^t \|u_n(s)\|_{1/2}^2\,\mathrm{d}s + \|u_0\|_{L^1}\right)^4\end{aligned}$$

for some $c > 0$. To control the $\|w_n\|_{3/2}\|w_n\|_1^2$ terms in the last two estimates we use the interpolation

$$\begin{aligned}&\int_0^t \|w_n(s)\|_{3/2}\|w_n(s)\|_1^2\,\mathrm{d}s \leq \int_0^t \|w_n(s)\|_{3/2}^2\|w_n(s)\|_{1/2}\,\mathrm{d}s\\&\quad\leq \frac{1}{5c_1c_2}\sup_{s\in[0,t]}\|w_n(s)\|_{1/2}^2 + c\left(\int_0^t \|w_n(s)\|_{3/2}^2\,\mathrm{d}s\right)^2\end{aligned}$$

for some $c > 0$. Recombining these estimates of I_0 and multiplying by 2, (5.8) becomes

$$\begin{aligned}&\sup_{s\in[0,t]}\|w_n(s)\|_{1/2}^2 + 2\int_0^t \|w_n(s)\|_{3/2}^2\,\mathrm{d}s\\&\leq a_1\int_0^t \|v_n(s)\|_1^4\,\mathrm{d}s + a_2\left(\int_0^t \|w_n(s)\|_{3/2}^2\,\mathrm{d}s\right)^2\\&\quad + a_3\int_0^t\left(\int_0^s \|u_n(r)\|_{1/2}^2\,\mathrm{d}r + \|u_0\|_{L^1}\right)^4\mathrm{d}s,\end{aligned}\tag{5.9}$$

where $a_1, a_2, a_3 > 0$ are independent of n and t. To simplify the last term we fix $c' > 0$ such that

$$\int_0^t \left(\int_0^s \|u_n(r)\|_{1/2}^2 \, \mathrm{d}r\right)^4 \mathrm{d}s \\ \leq c't\left(\int_0^t \|v_n(s)\|_{1/2}^2 \, \mathrm{d}s\right)^4 + c't^5 \sup_{s\in[0,t]} \|w_n\|_{1/2}^8.$$

Thus (5.9) becomes

$$\begin{aligned} &\sup_{s\in[0,t]} \|w_n(s)\|_{1/2}^2 + 2\int_0^t \|w_n(s)\|_{3/2}^2 \, \mathrm{d}s \\ &\leq a_1 \int_0^t \|v(s)\|_1^4 \, \mathrm{d}s + a_2 \left(\int_0^t \|w_n(s)\|_{3/2}^2 \, \mathrm{d}s\right)^2 + a_3 c' t \|u_0\|_{L^1}^4 \qquad (5.10) \\ &\quad + a_3 c' t \left(\int_0^t \|v(s)\|_{1/2}^2 \, \mathrm{d}s\right)^4 + a_3 c' t^5 \sup_{s\in[0,t]} \|w_n(s)\|_{1/2}^8, \end{aligned}$$

where we used the fact that $\|v_n(t)\|_\sigma$ is an increasing function of n for all $\sigma \geq 0$ and $t \in [0, T^n]$. In the next section we prove Lemma 5.6 that we now apply to (5.10). From that technical result it follows that there exists T, independent of n such that for all $t \in [0,T]$ and all n

$$\begin{aligned} &\sup_{s\in[0,t]} \|w_n(s)\|_{1/2}^2 + 2\int_0^t \|w_n(s)\|_{3/2}^2 \, \mathrm{d}s \\ &\leq a_1 \int_0^T \|v(s)\|_1^4 \, \mathrm{d}s + a_3 c' T \|u_0\|_{L^1}^4 + a_3 c' T \left(\int_0^T \|v(s)\|_{1/2}^2 \, \mathrm{d}s\right)^4 =: F(T). \end{aligned}$$

Now clearly F is independent of n, so using Lemma 5.5 we see that the sequence $(w_n)_{n=1}^\infty$ and hence $(u_n)_{n=1}^\infty$ is bounded in $L^\infty(0,T;H^{1/2})$ and $L^2(0,T;H^{3/2})$. Moreover one can easily show that the sequence of derivatives $(\partial_t u_n)_{n=1}^\infty$ is bounded in $L^2(0,T;H^{-1/2})$.

Passing to a subsequence using the Aubin-Lions lemma, in the usual way, we may assume that u_n converges to the required local weak solution in $L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$. □

We next prove uniqueness for the local solutions constructed above.

Proof (of Lemma 5.4). The following formal argument can be justified using Lemma 1.15 in the same way as Lemma 4.2 (see also the proofs of Proposition 6.5 and Lemma 6.13). Set $w = u - v$, then taking the product of the equation satisfied by w with $2\Lambda^1 w$ yields the estimate

$$\begin{aligned}\|w(t)\|_{1/2}^2 + 2\int_0^t \|w(s)\|_{3/2}^2 \,\mathrm{d}s \le c\int_0^t \|u(s)\|_{L^6}\|w(s)\|_1\|w(s)\|_{3/2}\,\mathrm{d}s \\ + c\int_0^t \|w(s)\|_{H^{1/2}}\|v(s)\|_{3/2}\|w(s)\|_{3/2}\,\mathrm{d}s.\end{aligned} \tag{5.11}$$

For the first term we use the interpolation $\|w\|_1^2 \le \|w\|_{1/2}\|w\|_{3/2}$ and Young's inequality to obtain

$$c\|u(s)\|_{L^6}\|w(s)\|_1\|w(s)\|_{3/2} \le c\|u(s)\|_{H^1}^4\|w(s)\|_{1/2}^2 + \|w(s)\|_{3/2}^2. \tag{5.12}$$

For the second we make use of Lemma 5.5 and the fact that $w(0) = 0$:

$$\begin{aligned}c\|w(s)\|_{H^{1/2}}\|v(s)\|_{3/2}\|w(s)\|_{3/2} \le c\|v(s)\|_{3/2}^2\|w(s)\|_{1/2}^2 + \|w(s)\|_{3/2}^2 \\ + c\|v(s)\|_{3/2}^2\left(\int_0^s \|w(r)\|_{1/2}\left(\|u(r)\|_{1/2} + \|v(r)\|_{1/2}\right)\,\mathrm{d}r\right)^2.\end{aligned} \tag{5.13}$$

The integral over $[0,t]$ of the last term in (5.13) is at most

$$c\left(\int_0^t \|v(s)\|_{3/2}^2\mathrm{d}s\right)\left(\int_0^t \|w(s)\|_{1/2}^2\mathrm{d}s\right)\left(2\int_0^t \|u(s)\|_{1/2}^2 + \|v(s)\|_{1/2}^2\mathrm{d}s\right).$$

As $u \in L^4(0,T;H^{1/2})$ and $v \in L^2(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$, this together with (5.11), (5.12) and (5.13) imply that

$$\|w(t)\|_{1/2}^2 \le \int_0^t G(s)\|w(s)\|_{1/2}^2\,\mathrm{d}s$$

for some $G \in L^1(0,T)$. A Gronwall inequality now implies that, since $\|w(0)\|_{1/2} = 0$, $\|w(t)\|_{1/2} = 0$ for all $t \in [0,T]$. Uniqueness now follows using Lemma 5.5.

□

5.2.2 Lower bounds on the existence times for the Galerkin approximations

In this section we prove the technical lemma that allowed us to deduce lower bounds on the existence times for the Galerkin approximations to solutions of the Burgers equations in $H^{1/2}(\mathbb{T}^3)$. We will also make use of it in the next chapter. The statement is more general than we really need for the applications in this work, but this does not add much to the complexity of the proof.

The hypotheses made in the lemma for the functions $f_n : [0, T_n) \to \mathbb{R}$ can be understood in the context of solutions to ODEs as follows. Suppose that T_n is the maximal existence time for a solution and that f_n is a function of the solution such that $f_n(t) \uparrow \infty$ as $t \uparrow T^*$ if and only if the solution blows up at time T^*; then either $T_n = T_0$ (some upper bound T_0, independent of n arising from the coefficients of the equations) or the solution cannot be extended beyond $T_n \in [0, T_0)$ and f_n blows up at T_n.

Lemma 5.6. *Let $(T_n)_{n=1}^{\infty}$ be a sequence of times $T_n \in (0, T_0]$ and let $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$ be two sequences of non-negative functions*

$$f_n, g_n : [0, T_n) \to \mathbb{R},$$

such that f_n is lower semi-continuous and g_n is measurable. In addition, suppose that for each n, either $T_n = T_0$ or $f_n \to \infty$ as $t \uparrow T_n$, T_n being the first such blowup time, and

$$\sup_{s\in[0,t]} f_n(s) + \int_0^t g_n(s)\,\mathrm{d}s \\ \le A(t)\left(\sup_{s\in[0,t]} f_n(s)\right)^{a+1} + B(t)\left(\int_0^t g_n(s)\,\mathrm{d}s\right)^{b+1} + C(t) \quad (5.14)$$

for all $t \in [0, T_n)$. Here a, $b \ge 0$ are constants and $A, B, C : [0, T_0] \to \mathbb{R}$ denote continuous non-decreasing, non-negative, functions such that

$$A(t)(C(t))^a \to 0 \text{ and } B(t)(C(t))^b \to 0$$

as $t \downarrow 0$. Then there exists $T \in (0, T_0]$ independent of n such that

$$f_n(t) + \int_0^t g_n(s)\,\mathrm{d}s \le 2C(T)$$

for all $t \in [0, T]$ and all n.

Proof. We define

$$\tau_n := \sup\left\{ t \in [0, T_n) : A(t) \left(\sup_{s\in[0,t]} f_n(s) \right)^a + B(t) \left(\int_0^t g_n(s)\,\mathrm{d}s \right)^b \leq \frac{1}{2} \right\}.$$

Note that $\tau_n > 0$ and that if $\tau_n < T_n$ then since A and B are continuous and f_n is lower semi-continuous,

$$A(\tau_n) \left(\sup_{s\in[0,\tau_n]} f_n(s) \right)^a + B(\tau_n) \left(\int_0^{\tau_n} g_n(s)\,\mathrm{d}s \right)^b = 1/2. \tag{5.15}$$

It follows by (5.14), that for all s, t such that $s < t \leq \tau_n$ we have

$$\sup_{r\in[0,s]} f_n(r) + \int_0^s g_n(r)\,\mathrm{d}r \leq \frac{1}{2} \sup_{r\in[0,s]} f_n(r) + \frac{1}{2} \int_0^s g_n(r)\,\mathrm{d}r + C(t).$$

In particular $f_n(s) + \int_0^s g_n(r)\,\mathrm{d}r \leq 2C(\tau_n)$ for $s < \tau_n$ or $s = \tau_n$ if $\tau_n < T_n$. It remains to find a lower bound on τ_n.

Let

$$T' := \sup\{t \in [0, T_0] : 2^a A(t)(C(t))^a + 2^b B(t)(C(t))^b < 1/2\},$$

then $T' > 0$ by continuity. Indeed, we assumed that

$$A(0)[C(0)]^a = B(0)[C(0)]^b = 0.$$

Suppose, for contradiction, that $T' > \tau_n$ then

$$\begin{aligned} A(\tau_n) \left(\sup_{s\in[0,\tau_n]} f_n(s) \right)^a &+ B(\tau_n) \left(\int_0^{\tau_n} g_n(s)\,\mathrm{d}s \right)^b \\ &\leq 2^a A(\tau_n)[C(\tau_n)]^a + 2^b B(\tau_n)[C(\tau_n)]^b < 1/2. \end{aligned}$$

If $\tau_n < T_n$, this is a contradiction to (5.15). Otherwise $\tau_n = T_n$, in which case $f_n(t) \leq 2C(T_n)$ for $t < T_n$, so no blowup occurs at time T_n, hence $\tau_n = T_n = T_0 \geq T'$ by hypothesis. This is also a contradiction to the supposition that $T' > \tau_n$.

In either case, it follows that $T_n \geq \tau_n \geq T'$ for all n. Fixing any $T < T'$ (or $T = T'$ if $T' \neq T_n$ for all n), we have $f_n(t) + \int_0^t g_n(r)\,\mathrm{d}r \leq 2C(T)$ for all $t \in [0, T]$, as required. $\square$

5.3 Conclusions

In the last chapter we adapted several well-known results about the Navier–Stokes equations and proved local well-posedness results for the diffusive Burgers equations in L^p for bounded domains and the whole spaces $\mathbb{R}^d$, for $p > d \geq 2$. These solutions can be uniquely extended for all time $t \geq 0$ by virtue of the maximum principle.

In this chapter we dealt with the more difficult case of the critical space $H^{1/2}(\mathbb{T}^3)$ and proved a global well-posedness result for arbitrary initial data in that space.

In both cases, many of the arguments that applied to the Navier–Stokes equations were not too difficult to adapt, however there were several technicalities to deal with and our careful treatment here is certainly merited.

Without the incompressibility constraint, we do not have the requisite energy estimates on the nonlinear term to obtain weak solutions for arbitrary L^2 initial data; therefore, regularity considerations alone would not have sufficed. Instead, we proved local existence in each of the spaces H^1, L^p and $H^{1/2}$.

The non-conservation of momentum complicated our estimates in several places. For a bounded domain with no-slip boundary condition this is not a significant difficulty, but obtaining domain-independent estimates in H^1 and L^p required some care. In $H^{1/2}(\mathbb{T}^3)$, we even required a separate estimate of the momentum growth caused by the nonlinear term, that made the estimates more difficult in that case.

Having proved several global well-posedness results for the Burgers equations using the Navier–Stokes equations as a reference, we are now in a position to give a similar treatment to a model we have found for the latter system. This is the focus of the next chapter. Therein we will discuss the magnetization variables formulation for the Navier–Stokes equations (including a careful analysis of equivalence in a weak setting) and the aforementioned model system that we will prove is globally well-posed, following the methods applicable to the Burgers equations.

Chapter 6

The Navier–Stokes equations: magnetization variables and a model system

6.1 Introduction

As we noted at the end of Chapter 3 in (3.69), the Navier–Stokes equations can be reformulated as a system without an explicit pressure term using the so-called *magnetization variable*. In this formulation, previously discussed by Montgomery-Smith and Pokorný (2001), incompressibility is enforced explicitly via a Leray projection. The magnetization variables are more well-known in the study of the Euler equations, see Chorin (1994).

The reformulated system is as follows, where u can be thought of as the velocity in the classical formulation and $w = (w^1, w^2, w^3)$ are the magnetization variables:

$$w_t + (u \cdot \nabla)w + (\nabla u)^\top w - \Delta w = 0 \tag{6.1}$$

$$u = \mathbb{P}w. \tag{6.2}$$

Several names have been used for these variables in the literature, including "velicity" and "impulse variables", however the term "magnetization variables" seems most widely used. The reason behind this nomenclature is alluded to on pp. 20 of Chorin (1994) (see also Buttke (1993)); the idea being that w corresponds to the magnetization in an analogy between fluid mechanics and magnetostatics. We will give a brief (and somewhat heuristic) discussion of this correspondence at the end of this section.

In this chapter, following Pooley (2016), we will give further discussion of the derivation of this system and prove that it is equivalent to the Navier–Stokes system for solutions in $H^{1/2}(\mathbb{T}^3)$. We will also prove a partial equivalence result for L^2 weak solutions. Following this, we present a slight simplification of this system, which we will refer to as the *model system*, that admits a maximum principle. For this model system we will follow the analysis used in Chapter 5 for the Burgers equations to prove a global well-posedness result in $H^{1/2}(\mathbb{T}^3)$.

To be more specific, in Section 6.2 we will show that for a classical solution w of (6.1), $u = \mathbb{P}w$ is a solution of the Navier–Stokes equations for some p. Conversely for a classical solution of the Navier–Stokes equations there exists at least one classical solution w of (6.1) such that $\mathbb{P}w = u$.

In the context of weak solutions on $\mathbb{T}^3$ (which are defined below), we will show that for a weak solution $w \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ the projection $\mathbb{P}w$ is a weak solution of the Navier–Stokes equations. As a partial converse we show that if $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ is a weak solution of the Navier–Stokes equations then any $w \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ such that $\mathbb{P}w = u$ for all $t \in [0,T)$ satisfies the weak form of (6.1) when tested against divergence-free functions, but possibly not in full generality.

We then consider (6.1) as a linear system for a fixed velocity

$$u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$$

(without the requirement that $u = \mathbb{P}w$, necessarily) and show that for $w_0 \in H^{1/2}$ there exists a unique solution

$$w \in C(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2}).$$

To see that $u = \mathbb{P}w$ holds if u is a weak solution of the Navier–Stokes equations, we will observe that u and $\mathbb{P}w$ are both weak solutions of a certain Dirichlet problem that has a unique weak solution (subject to compatibility of the initial data: $\mathbb{P}w_0 = u_0$). It follows that the two formulations are equivalent, in the context of weak solutions with the stronger regularity

$u, w \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$.

In Section 6.3 we prove global well-posedness in $H^{1/2}(\mathbb{T}^3)$ for the system

$$w_t + ((\mathbb{P}w)\cdot\nabla)w + \frac{1}{2}\nabla|w|^2 - \Delta w = 0,$$

which is a simplification obtainted from (6.1) by replacing $u = \mathbb{P}w$ with w in the term $(\nabla u)^\top w$. This analysis will be a direct analogue of our treatment of the Burgers equations in Chapter 5 (see also Pooley and Robinson (2016a)).

As in the previous chapter we will denote by Λ^s, the fractional derivative

$$\Lambda^s f(x) \coloneqq \sum_{k\in\mathbb{Z}^3} |k|^s \hat{f}_k \mathrm{e}^{\mathrm{i}k\cdot x} \in L^2(\mathbb{T}^3),$$

and by $\|\cdot\|_s$ the seminorm

$$\|f\|_s \coloneqq \left(\sum_{k\in\mathbb{Z}^3} |k|^{2s}|\hat{f}_k|^2\right)^{1/2}.$$

To conclude this section we remark on two other models for the Navier–Stokes equations that have recently been discussed in the literature, and give an explanation of the origin of the term “magnetization variables”.

Two recent studies of other models

The study of models of the Navier–Stokes equations, via the analysis of a reformulation with modified nonlinearity, is in the spirit of other recent work. For example, Chae (2015) discusses the equivalence between the Navier–Stokes equations and the system

$$\begin{cases} u_t + R\times R\times(u\times\omega) = \Delta u, \\ \omega = \nabla\times u \end{cases}$$

where

$$R\times u \coloneqq (R_2u_3 - R_3u_2, R_3u_1 - R_1u_3, R_1u_2 - R_2u_1)$$

is a combination of the Riesz transforms R_1, R_2 and R_3 on $\mathbb{R}^3$ applied to $u = (u_1, u_2, u_3)$. He then shows that the simplified system

$$\begin{cases} u_t + R \times (u \times \omega) = \Delta u, \\ \omega = \nabla \times u \end{cases}$$

is globally well-posed, in the sense of weak solutions

$$u \in C([0,T); H^m(\mathbb{R}^3)) \cap L^2(0,T; H^{m+1}(\mathbb{R}^3))$$

for $m > 5/2$.

In contrast, it is shown by Tao (2014), that there exists an "averaged" version of the classical nonlinear term such that the modified system admits a smooth solution that blows up in finite time.

Remarks on the term "magnetization variables"

Following a comment by Chorin (1994) and adding a few details on magnetostatics from Jackson (1975), we now give a brief explanation for the use of the term "magnetization variables".

We first note that there is a well-known analogy between fluid mechanics and magnetostatics that is furnished by the fact that a velocity field u can be recovered from its corresponding vorticity ω, by solving the relations

$$\nabla \times u = \omega, \text{ and } \nabla \cdot u = 0,$$

just as for a given current density J, the induced magnetic field B satisfies Ampère's law:

$$\nabla \times B = cJ, \text{ and } \nabla \cdot B = 0,$$

for some absolute constant $c > 0$. This correspondence can also be thought of in terms of the *Biot–Savart law* (see the discussions of Biot–Savart in Robinson et al. (2016) and Jackson (1975) for fluids and magnetostatics respectively).

To make this framework a little more practical for problems involving magnetic material, one can let J denote only the macroscopic currents and introduce the *magnetic moment density* or *magnetization* M that models microscopic[1] effects. The magnetization effectively contributes the magnetic field that would be induced by a current

$$J_M := c\nabla \times M.$$

That is to say, B is taken to be the solution of

$$\nabla \times B = J_M + cJ, \ \nabla \cdot B = 0.$$

Hence, in the case of vanishing macroscopic current (i.e. $J = 0$) we have

$$\nabla \times B = c\nabla \times M \text{ and } \nabla \cdot B = 0$$

which can be restated using the Helmholtz decomposition as

$$B = c\mathbb{P}M.$$

Therefore w corresponds to the magnetization M, in the analogy, where u corresponds to B, and ω corresponds to J_M.

6.2 The magnetization variables formulation: derivation and equivalence

6.2.1 Classical solutions

The following propositions show that the Navier–Stokes equations and the magnetization variables formulation (6.1, 6.2) are equivalent in the context of classical solutions on the interior of a domain $\Omega \subseteq \mathbb{R}^3$ (or on the torus $\mathbb{T}^3$). The manipulations in the proofs are similar to the derivation of the Weber formula for the Euler equations in Chapter 3 (see also Constantin (2000) or Pooley and Robinson (2016b)).

Proposition 6.1. *If $u, w \in C^1([0,T]; C^2(\Omega))$ satisfy (6.1) and $u = \mathbb{P}w$ then there exists $p \in C([0,T]; C^1(\Omega))$ such that (u,p) is a solution of the Navier–Stokes equations.*

[1] By microscopic effects, we mean intra-atomic currents, for example. Such currents can be very difficult to measure, hence the need to model them by the macroscopic variable M.

Proof. By the Helmholtz decomposition there exists $q \in C^1([0,T];C^3(\Omega))$ such that

$$u = w - \nabla q.$$

It is clear that u is divergence free so we must prove that it satisfies

$$u_t + (u\cdot\nabla)u - \Delta u + \nabla p = 0.$$

Indeed we have

$$\begin{aligned}
u_t + (u\cdot\nabla)u - \Delta u &= w_t + (u\cdot\nabla)w - \Delta w - \nabla(q_t - \Delta q) - (u\cdot\nabla)\nabla q \\
&= w_t + (u\cdot\nabla)w - \Delta w \\
&\quad - \nabla(q_t - \Delta q + (u\cdot\nabla)q + \tfrac{1}{2}|u|^2) + (\nabla u)^\top w \\
&= -\nabla p,
\end{aligned}$$

where $p \coloneqq (q_t - \Delta q + (u\cdot\nabla)q + \frac{1}{2}|u|^2)$. In the second line we used the commutation relation (see Constantin (2000))

$$\begin{aligned}
(u\cdot\nabla)\nabla q = \nabla[(u\cdot\nabla)q] - (\nabla u)^\top\nabla q &= \nabla[(u\cdot\nabla)q] - (\nabla u)^\top\nabla(w-u) \\
&= \nabla[(u\cdot\nabla)q] + \frac{1}{2}\nabla|u|^2 - (\nabla u)^\top w.
\end{aligned}$$

□

Proposition 6.2. *Suppose that $u \in C^1([0,T];C^2)$ and $p \in C([0,T];C^1)$ are a classical solution of the Navier–Stokes equations. If $w_0 \in C^2(\Omega)$ such that $\mathbb{P}w_0 = u(0)$, there exists a unique $w \in C^1([0,T];C^2(\Omega))$ such that u, w satisfy (6.1) and $u = \mathbb{P}w$.*

Proof. By standard techniques for parabolic PDEs (see Chapter 7 of Evans (2010), for example) there exists a unique $q \in C^1([0,T];C^3(\Omega))$ such that

$$\partial_t q + (u\cdot\nabla)q - \Delta q = p - \frac{1}{2}|u|^2$$

and $q(t,x) = 0$ for all $(t,x) \in (\{0\}\times\Omega)\cup([0,T)\times\partial\Omega)$. If we set $w \coloneqq u + \nabla q$

then $u = \mathbb{P}(w)$ and

$$\begin{aligned} w_t + u\cdot\nabla w + (\nabla u)^\top w - \Delta w &= u_t + (u\cdot\nabla)u + \frac{1}{2}\nabla|u|^2 - \Delta u \\ &\quad +\nabla q_t + (u\cdot\nabla)\nabla q + (\nabla u)^\top \nabla q - \nabla\Delta q \\ &= \nabla\left(-p + \frac{1}{2}|u|^2 + q_t + (u\cdot\nabla)q - \Delta q\right) = 0. \end{aligned}$$

Hence there exists $w \in C^1([0,T];C^2(\Omega))$ such that u, w satisfy (6.1) and (6.2).

Uniqueness follows from the fact that any two solutions w_1 and w_2 differ only by a gradient $\nabla\tilde{q}$ for some $\tilde{q}$ that satisfies

$$\partial_t\tilde{q} + (u\cdot\nabla)\tilde{q} - \Delta\tilde{q} = h(t);\ \tilde{q}(0,x) = C$$

for some function h that is independent of x, and some constant C. Hence $\tilde{q}$ depends only on time, and $\nabla q \equiv 0$. □

6.2.2 Partial equivalence for weak solutions

Proposition 6.1 can be strengthened to apply to weak solutions of (6.1) and (6.2). We say that $w \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ is a weak solution of (6.1) for initial data $w_0 \in L^2(\mathbb{T}^3)$ if for all $\phi \in C_c^\infty([0,T)\times\mathbb{T}^3)$ and all $t \in [0,T)$

$$\begin{aligned} (w(t),\phi(t))_{L^2} + \int_0^t ((\mathbb{P}w\cdot\nabla)w + (\nabla\mathbb{P}w)^\top w, \phi)_{L^2} + (\nabla w, \nabla\phi)_{L^2}\,\mathrm{d}s \\ = (w_0,\phi(0))_{L^2} + \int_0^t (w(s),\partial_t\phi(s))_{L^2}\,\mathrm{d}s. \end{aligned} \tag{6.3}$$

As in the previous chapter, we call a weak solution w an $H^{1/2}$ solution if it has the additional regularity $w \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$.

For the Navier–Stokes equations recall from Definition 2.3 that we say $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ is a weak solution corresponding to the initial data $u_0 \in L^2(\mathbb{T}^3)$ if

$$\begin{aligned} (u(t),\psi(t))_{L^2} + \int_0^t ((u\cdot\nabla)u,\psi)_{L^2} + (\nabla u,\nabla\psi)_{L^2}\mathrm{d}s \\ = (u_0,\psi(0))_{L^2} + \int_0^t (u(s),\partial_t\phi(s))_{L^2}\,\mathrm{d}s \end{aligned} \tag{6.4}$$

for all $t \in [0,T)$ for any divergence-free test functions $\psi \in C_c^\infty([0,T) \times \mathbb{T}^3)$ with $\nabla \cdot \psi \equiv 0$.

Proposition 6.3. *Suppose that $w \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ is a weak solution of (6.1) for initial data $w_0 \in L^2(\mathbb{T}^3)$. Then $u := \mathbb{P}w$ is a weak solution of the Navier–Stokes equations for initial data $u_0 = \mathbb{P}w_0$.*

Proof. The main ingredient in the proof is the fact that if $v \in H^1(\mathbb{T}^3)$ then for all $\psi \in C_c^\infty(\mathbb{T}^3)$ with $\nabla \cdot \psi = 0$ we have

$$((\mathbb{P}v \cdot \nabla)v + (\nabla \mathbb{P}v)^\top v, \psi)_{L^2} = ((\mathbb{P}v \cdot \nabla)\mathbb{P}v, \psi)_{L^2}. \tag{6.5}$$

For $v \in C^\infty(\mathbb{T}^3)$ we can argue as we did in the proof of Proposition 6.1 to prove (6.5). For $v \in H^1(\mathbb{T}^3)$ we can consider a sequence of approximations in $C^\infty(\mathbb{T}^3)$ and show that (6.5) passes to the limit.

Fixing a divergence-free test function $\phi \in C_c^\infty([0,T) \times \mathbb{T}^3)$ and using the symmetry of the Leray projection, (6.3) becomes:

$$\begin{aligned}(\mathbb{P}w(t), \phi(t))_{L^2} + \int_0^t ((\mathbb{P}w \cdot \nabla)w + (\nabla \mathbb{P}w)^\top w, \phi)_{L^2} + (\nabla \mathbb{P}w, \nabla\phi)_{L^2} \mathrm{d}s \\ = (\mathbb{P}w_0, \phi(0))_{L^2} + \int_0^t (\mathbb{P}w(s), \partial_t\phi(s))_{L^2}\,\mathrm{d}s.\end{aligned}$$

By (6.5) we can set $u := \mathbb{P}w$ to obtain the required solution of (6.4). $\square$

Another consequence of (6.5) is the following partial converse.

Corollary 6.4. *If $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ is a weak solution of the Navier–Stokes equations then, for any $w \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ such that $\mathbb{P}w = u$, w satisfies (6.3) for all test functions $\phi \in C_c^\infty([0,T) \times \mathbb{T}^3)$ that are divergence free.*

Note that this does not imply that w is a weak solution of (6.1), since in the definition we allowed test functions with non-zero divergence.

6.2.3 Well-posedness of the linear system

In Section 6.2.4 we will show that a weak solution of the Navier–Stokes equations with the regularity $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2}))$ corresponds to a unique $H^{1/2}$ solution w of (6.1) such that $\mathbb{P}w = u$, subject to fixing w_0 with $\mathbb{P}w_0 = u_0$. This will follow from the main result of this section, which gives local well-posedness for the linear system

$$w_t + (u \cdot \nabla)w + (\nabla u)^\top w - \Delta w = 0 \tag{6.6}$$

i.e. (6.1), where $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$ is fixed. In this section we do not assume that u solves the Navier–Stokes equations.

For $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$, we say that w is an $H^{1/2}$ solution of (6.6) if $w \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$ and

$$\begin{aligned}(w(t),\phi(t)) &+ \int_0^t ((u\cdot\nabla)w(s) + (\nabla u)^\top w(s), \phi(s))_{L^2}\,\mathrm{d}s \\ &= (w_0,\phi(0))_{L^2} + \int_0^t (w(s),\partial_t\phi(s))_{L^2}\,\mathrm{d}s - \int_0^t (\nabla w(s), \nabla\phi(s))_{L^2}\,\mathrm{d}s \end{aligned}\tag{6.7}$$

for all $t \in [0,T)$ and all $\phi \in C_c^\infty([0,T) \times \mathbb{T}^3)$.

As in the case of classical solutions, there is a one-to-one correspondence between $H^{1/2}$ solutions of (6.3) and (6.4) (after fixing a correspondence between the initial data). We will prove this using the uniqueness from the following proposition and an analogous argument in the next section.

Proposition 6.5. *Fix $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$ and $w_0 \in H^{1/2}$. There exists a unique weak solution $w \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$ to (6.6).*

In the proof of Proposition 6.5 we will consider approximate solutions $w_n = P_n w_n$ that solve

$$\begin{cases} \partial_t w_n + P_n[(P_n u \cdot \nabla)w_n + (\nabla P_n u)^\top w_n] - \Delta w_n = 0, \\ w_n(0) = P_n w_0, \end{cases} \tag{6.8}$$

which exists on a maximal time interval $[0,T_n)$, because this is a finite-dimensional system of Lipschitz ODEs. As in Chapter 5 we will need the following lemma, so that we can estimate the evolution of the momentum of w, in order to control inhomogeneous norms of w_n. The proof is similar to the proof of Lemma 5.5.

Lemma 6.6. *If w_n solves (6.8), for some n, then for any $s > 0$ there exists*

$c_s > 0$ such that

$$\|w_n(t)\|_{H^s} \le c_s \|w_n(t)\|_s + c_s \int_0^t \|w_n\|_{1/2} \|u\|_{1/2} + c_s \|w_n(0)\|_{L^1} \tag{6.9}$$

for all $t \in [0, T_n)$.

Proof. To save notation set $w := w_n$. The zeroth Fourier coefficient of w satisfies

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \hat{w}_0 \right| = \frac{1}{(2\pi)^{3/2}} \left| \sum_{|k| \le n} \hat{u}_{-k} \cdot k \hat{w}_k + k \hat{u}_k \cdot \hat{w}_{-k} \right| \le \frac{2}{(2\pi)^{3/2}} \|w\|_{1/2} \|u\|_{1/2},$$

so

$$|\hat{w}_0(t)| \le |\hat{w}_0(0)| + \frac{2}{(2\pi)^{3/2}} \int_0^t \|w\|_{1/2} \|u\|_{1/2}$$

for all $t \in [0, T_n]$. The result now follows because

$$\|w(t)\|_{H^s} \le c \|w(t)\|_s + c |\hat{w}_0(t)|,$$

for some c that depends only on s, and since

$$(2\pi)^{3/2} |\hat{w}_0(0)| = \left| \int_{\mathbb{T}^3} w(x, 0) \, \mathrm{d}x \right| \le \|w(0)\|_{L^1}. \qquad \square$$

Note that we do not assume that $\nabla \cdot u = 0$ in Proposition 6.5, because we would not gain much by such an assumption. Indeed the term $(\nabla u)^\top w$ may still break momentum conservation, even if u is divergence free. This is in contrast with the Navier–Stokes equations, where the corresponding nonlinear term is $(\nabla \mathbb{P} w)^\top w$, for which the integral over $\mathbb{T}^3$ vanishes.

Proof (of Proposition 6.5). Taking the L^2 product of (6.8) with Λw_n yields

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|w_n\|_{1/2}^2 + \|w_n\|_{3/2}^2 \\
&\qquad \le c \|P_n u\|_{L^6} \|\nabla w_n\|_{L^3} \|w_n\|_1 + c \|\nabla P_n u\|_{L^3} \|w_n\|_{L^3} \|\Lambda w_n\|_{L^3} \\
&\qquad \quad \le c \|u\|_{H^1} \|w_n\|_{1/2}^{1/2} \|w_n\|_{3/2}^{3/2} + c \|u\|_{3/2} \|w_n\|_{H^{1/2}} \|w_n\|_{3/2} \\
&\qquad \qquad \le c \|u\|_{H^1}^4 \|w_n\|_{1/2}^2 + c \|u\|_{3/2}^2 \|w_n\|_{H^{1/2}}^2 + \frac{1}{2} \|w_n\|_{3/2}^2,
\end{aligned} \tag{6.10}$$

where we have used the Sobolev embeddings $H^1 \hookrightarrow L^6$, $H^{1/2} \hookrightarrow L^3$ and the interpolation $\|w_n\|_1 \le \|w_n\|_{1/2}^{1/2}\|w_n\|_{3/2}^{1/2}$.

By Lemma 6.6 and the embedding $H^{1/2}(\mathbb{T}^3) \hookrightarrow L^1(\mathbb{T}^3)$ we have

$$\|w_n(t)\|_{H^{1/2}}^2 \le c\|w_n\|_{1/2}^2 + c\left(\int_0^t \|w_n(s)\|_{1/2}\|u(s)\|_{1/2}\mathrm{d}s\right)^2 + c\|w_n(0)\|_{H^{1/2}}^2.$$

Hence, after integrating, (6.10) becomes

$$\begin{aligned}\|w_n(t)\|_{1/2}^2 + \int_0^t \|w_n(s)\|_{3/2}^2\mathrm{d}s \\ &\le c\int_0^t (\|u(s)\|_{H^1}^4 + \|u(s)\|_{3/2}^2)\|w_n(s)\|_{1/2}^2\mathrm{d}s \\ &+ c\int_0^t \|u\|_{3/2}^2\mathrm{d}s\left(\int_0^t \|w_n(s)\|_{1/2}\|u(s)\|_{1/2}\mathrm{d}s\right)^2 \\ &+ \|w_n(0)\|_{H^{1/2}}^2\left(1 + c\int_0^t \|u\|_{3/2}^2\mathrm{d}s\right). \quad (6.11)\end{aligned}$$

It follows that for all $t \in [0, T_n)$,

$$\begin{aligned}\sup_{s\in[0,t]} \|w_n(t)\|_{1/2}^2 \le c \sup_{s\in[0,t]} \|w_n(s)\|_{1/2}^2 \left[\int_0^t (\|u(s)\|_{H^1}^4 + \|u(s)\|_{3/2}^2)\mathrm{d}s\right. \\ \left. + t\int_0^t \|u\|_{3/2}^2\mathrm{d}s \int_0^t \|u\|_{1/2}^2\mathrm{d}s\right] + \|w_0\|_{H^{1/2}}^2\left(1 + c\int_0^t \|u\|_{3/2}^2\mathrm{d}s\right). \quad (6.12)\end{aligned}$$

Hence $\|w_n(t)\|_{1/2}$ is bounded on $[0, T']$, given by

$$\begin{aligned}T' := \frac{1}{2}\sup\Big\{t \in [0, T) : c\int_0^t (\|u(s)\|_{H^1}^4 + \|u(s)\|_{3/2}^2)\,\mathrm{d}s \\ + ct\int_0^t \|u\|_{3/2}^2\mathrm{d}s \int_0^t \|u\|_{1/2}^2\,\mathrm{d}s < 1/2\Big\}\end{aligned}$$

where c is the absolute constant from (6.12). This can also be deduced by applying Lemma 5.6 to (6.12).

By (6.11), we see that $w_n \in L^\infty(0, T'; H^{1/2}) \cap L^2(0, T'; H^{3/2})$ is uniformly bounded. Integrating (6.8) against a function $v \in H^{1/2}$ we also see that $\partial_t w_n$ is uniformly bounded in $L^2(0, T'; H^{-1/2})$. By the Aubin–Lions lemma we deduce that there exists a subsequence of $(w_n)_{n=1}^\infty$ converging to a limit $w \in L^\infty(0, T'; H^{1/2}) \cap L^2(0, T'; H^{3/2})$ with $\partial_t w \in L^2(0, T'; H^{-1/2})$, and

$$\begin{aligned}\int_0^\infty ((u\cdot\nabla)w(s) + (\nabla u)^\top w(s), \phi(s))_{L^2}\,\mathrm{d}s \\ = (w_0, \phi(0))_{L^2} + \int_0^\infty (w(s), \partial_t\phi(s))_{L^2}\,\mathrm{d}s - \int_0^\infty (\nabla w(s), \nabla\phi(s))_{L^2}\,\mathrm{d}s\end{aligned} \tag{6.13}$$

for any $\phi \in C^\infty([0,T)\times\mathbb{T}^3)$. By redefining the limit on a set of times with zero measure we can obtain an $H^{1/2}$ solution to (6.6) that is a weakly continuous function of time into L^2 (see Galdi (2000)). Moreover since $\partial_t w \in L^2(0,T';H^{-1/2})$ and $w \in L^2(0,T';H^{3/2})$, $w \in C([0,T'];H^{1/2})$ (see Lemma 1.12).

To construct a solution on $[0,T) \supseteq [0,T']$ we use the definition of T' and the fact that

$$\int_0^T (\|u(s)\|_{H^1}^4 + \|u(s)\|_{3/2}^2)\,\mathrm{d}s + cT\int_0^T \|u\|_{3/2}^2\,\mathrm{d}s \int_0^T \|u\|_{1/2}^2\,\mathrm{d}s < \infty$$

to show that for all $\varepsilon > 0$, the existence argument above can be repeated a finite number of times to construct a solution

$$w \in C([0,T-\varepsilon];H^{1/2}) \cap L^2(0,T-\varepsilon;H^{3/2})$$

such that $\partial_t w \in L^2(0,T-\varepsilon;H^{-1/2})$.

We next prove uniqueness for such solutions w. Indeed, (6.7) is linear in w so it suffices to consider the case $w_0 = 0$. Proceeding formally as we did in the derivation of (6.12) yields

$$\begin{aligned}\sup_{t\in[t_1,t_2]} \|w\|_{1/2}^2 + \int_{t_1}^{t_2} \|w(s)\|_{3/2}^2\,\mathrm{d}s \le A(t_1,t_2) \sup_{t\in[t_1,t_2]} \|w\|_{1/2}^2 \\ + \|w(t_1)\|_{H^{1/2}}^2 \left(1 + c\int_{t_1}^{t_2} \|u(s)\|_{3/2}^2\,\mathrm{d}s\right)\end{aligned} \tag{6.14}$$

for any $[t_1,t_2] \subseteq [0,T)$, where

$$\begin{aligned}A(t_1,t_2) \coloneqq c\int_{t_1}^{t_2} (\|u(s)\|_{H^1}^4 + \|u(s)\|_{3/2}^2)\,\mathrm{d}s \\ + c(t_2-t_1)\int_{t_1}^{t_2} \|u(s)\|_{H^{3/2}}^2\,\mathrm{d}s \int_{t_1}^{t_2} \|u(s)\|_{1/2}^2\,\mathrm{d}s.\end{aligned}$$

Thus, if $[t_1, t_2] \subseteq [0, T)$ such that $A(t_1, t_2) < \frac{1}{2}$ then

$$\|w(t_2)\|_{1/2}^2 \leq 2\|w(t_1)\|_{H^{1/2}}^2 \left(1 + c\int_{t_1}^{t_2} \|u(s)\|_{3/2}^2 \,\mathrm{d}s\right). \tag{6.15}$$

It follows that if $w(t) = 0$ for $t \in [0, S]$ then it also vanishes on $[S, S+\varepsilon]$ where $A(S, S+\varepsilon) = 1/3$. Since $A(0, T) < \infty$ we can iterate this argument to see that $w \equiv 0$ if $w_0 = 0$.

As in Lemma 4.2, we can justify (6.14) by considering a sequence of smooth test functions $\psi_n \in L^2(0, T; H^1)$ converging to $\Lambda^{1/2}w$ with respect to the norms of $L^2(0, S; H^1)$ and $C([0, S]; L^2)$, and $\partial_t \psi_n \to \partial_t \Lambda^{1/2} w$ in $L^2(0, S; H^{-1})$, for any $S \in [0, T)$. Such a sequence exists by Lemma 1.15. Using $\phi_n = \Lambda^{1/2}\psi_n$ as test functions[2] in (6.7), we see that

$$\begin{aligned}
(\Lambda^{1/2}w_0, \psi_n(0))_{L^2} &+ \int_0^t (\Lambda^{1/2}w(s), \partial_t\psi_n)\,\mathrm{d}s \\
&\to \|w_0\|_{1/2}^2 + \int_0^t \langle \Lambda^{1/2}w(s), \partial_t\Lambda^{1/2}w(s)\rangle_{H^1\times H^{-1}}\,\mathrm{d}s \\
&= \frac{1}{2}\|w(t)\|_{1/2}^2 - \frac{1}{2}\|w_0\|_{1/2}^2,
\end{aligned} \tag{6.16}$$

as $n \to \infty$, for all $t \in [0, S)$. For the other terms in (6.7) we use

$$\begin{aligned}
\int_0^t ((u\cdot\nabla)w(s) &+ (\nabla u)^\top w(s), \Lambda^{1/2}\psi_n(s))_{L^2} + (\Lambda^{3/2}w(s), \Lambda^1\psi_n(s))_{L^2}\,\mathrm{d}s \\
&\leq c\int_0^t \|u\|_{H^1}\|\nabla w\|_1\|\psi_n\|_1 + \|u\|_{3/2}\|w\|_{1/2}\|\psi_n\|_1 + \|w\|_{3/2}\|\psi_n\|_1\,\mathrm{d}s,
\end{aligned} \tag{6.17}$$

then proceed as in the formal calculation. To fully justify this, we also need to check that Lemma 6.6 can be strengthened to apply to weak solutions, but this can be proved by setting $\phi \equiv 1$ in (6.7) and considering Fourier expansions. Hence (6.14) also holds for solutions of (6.7), and so w is the unique solution on $[0, S]$ for any $S \in [0, T)$, hence on $[0, T)$. We have now proved Proposition 6.5. □

[2]Note that as we are working in a compact domain ($\mathbb{T}^3$) we only need $\Lambda^{1/2}\psi_n$ to be smooth. On a non-compact domain, ϕ_n might not be compactly supported even if ψ_n is.

6.2.4 Equivalence for $H^{1/2}$ solutions

In the previous section we proved well-posedness for $H^{1/2}$ solutions of (6.6), given a fixed $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$. In this section we will show that if we also assume that u is a weak solution of the Navier–Stokes equations then $u = \mathbb{P}w$, subject to choosing the initial data w_0 such that $\mathbb{P}w_0 = u_0$.

Proposition 6.7. *If $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$ is a weak solution of the Navier–Stokes equations and $\mathbb{P}w_0 = u_0$ then the corresponding solution w of (6.7) satisfies $\mathbb{P}w = u$, i.e. (u,w) satisfy (6.1) and (6.2) in a weak sense.*

Proof. Fix a weak solution $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$ of the Navier–Stokes equations and let $w \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$ be the corresponding solution of (6.7). Then $v := \mathbb{P}w$ satisfies

$$\begin{aligned}(v(t),\phi(t))_{L^2} &+ \int_0^t ((u\cdot\nabla)v(s) + (\nabla u)^\top v(s),\phi(s))_{L^2}\,\mathrm{d}s \\ &= (v_0,\phi(0))_{L^2} + \int_0^t (v(s),\partial_t\phi(s))\,\mathrm{d}s - \int_0^t (\nabla v(s),\nabla\phi(s))_{L^2}\mathrm{d}s \qquad (6.18)\end{aligned}$$

for all $t \in [0,T)$ and all $\phi \in C_c^\infty([0,T)\times\mathbb{T}^3)$ such that $\nabla\cdot\phi = 0$. Indeed, if u and w are smooth we can write $v = w - \nabla q$ and treat the nonlinear terms as follows:

$$\begin{aligned}((u\cdot\nabla)w + (\nabla u)^\top w,\phi) &= ((u\cdot\nabla)v + (\nabla u)^\top v + (u\cdot\nabla)\nabla q + (\nabla u)^\top\nabla q,\phi) \\ &= ((u\cdot\nabla)v + (\nabla u)^\top v,\phi)\,.\end{aligned}$$

For weak solutions, we can justify this with sequences of approximations (see the proof of Proposition 6.3).

Notice that the equations (6.7) and (6.18) differ only in the space of allowed test functions. Now by a uniqueness argument analogous to the one above for (6.7), we see that a divergence-free $H^{1/2}$ solution of (6.18), $v \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$, is unique.

From the hypothesis that u solves the Navier–Stokes equations, we deduce that u solves (6.18), indeed using the substitution $v = u$ the nonlinear term becomes

$$((u\cdot\nabla)u + \tfrac{1}{2}\nabla|u|^2,\phi)_{L^2} = ((u\cdot\nabla)u,\phi)_{L^2},$$

since $\nabla \cdot \phi = 0$. It therefore follows from the uniqueness of $H^{1/2}$ solutions of (6.18) that $u = v = \mathbb{P}w$ as claimed. $\square$

6.3 Global well-posedness for a model system

So far, we have considered the magnetization variables in the Navier–Stokes equations and proved the equivalence of the formulations for sufficiently regular weak solutions. Due to this equivalence, we do not expect to the reformulation to immediately yield new information about the Navier–Stokes equations. However, as mentioned above we have found a slight modification of the reformulation (which we call the "model system"). The proof of global well-posedness for this model system is the subject of this section.

Recall that the equations satisfied by the magnetization variables is

$$w_t + ((\mathbb{P}w) \cdot \nabla)w + (\nabla \mathbb{P}w)^\top w - \Delta w = 0.$$

We will consider the following simplification, obtained by replacing $\mathbb{P}w$ with w in only the second nonlinear term:

$$w_t + ((\mathbb{P}w) \cdot \nabla)w + \frac{1}{2}\nabla |w|^2 - \Delta w = 0. \tag{6.19}$$

In the context of finding estimates on solutions, (6.19) bears a strong resemblance to the Burgers equations that we studied in Chapters 4 and 5. Therein, we showed that for initial data in $w_0 \in H^{1/2}(\mathbb{T}^3)$, the latter system admits a unique $H^{1/2}$ solution which is classical for $t > 0$. We will show that similar methods apply to (6.19), which are closer to the Navier–Stokes equations in the sense that the nonlinear terms would have to be altered more significantly to obtain the Burgers equations. Moreover we will see that unlike solutions of the Burgers equations, solutions of (6.19) have constant momentum – a property shared with solutions of the Navier–Stokes equations.

As we did in the previous chapter, we divide the proof of well-posedness for (6.19) into two parts: first we prove the global well-posedness of weak solutions for initial data $w_0 \in H^1$; then show a local well-posedness result for $w_0 \in H^{1/2}$ that combines with the H^1 result to give global well-posedness in this case.

As for the Burgers equations (6.19) admits a maximum principle, which we have adapted from Kiselev and Ladyzhenskaya (1957):

Lemma 6.8. *If u is a classical solution of (6.19) on a time interval $[a, b]$ then*

$$\sup_{t \in [a,b]} \|w(t)\|_{L^\infty} \leq \|w(a)\|_{L^\infty}. \tag{6.20}$$

Proof. Fix $\alpha > 0$ and let $v(t, x) := \mathrm{e}^{-\alpha t} w(x, t)$ for all $x \in \mathbb{T}^3$. Then $|v|^2$ satisfies the equation

$$\frac{\partial}{\partial t}|v|^2 + 2\alpha |v|^2 + \nabla |v|^2 w + (\mathbb{P}w) \cdot \nabla |v|^2 - 2v \cdot \Delta v = 0. \tag{6.21}$$

Since $2v \cdot \Delta v = \Delta |v|^2 - 2|\nabla v|^2$ we see that if $|v|^2$ has a local maximum at $(x, t) \in (a, b] \times \mathbb{T}^3$ then the left-hand side of (6.21) is positive unless $|v(x, t)| = 0$. Hence

$$\|w(t)\|_{L^\infty} \leq \mathrm{e}^{\alpha t} \|w(a)\|_{L^\infty}.$$

Now (6.20) follows because $\alpha > 0$ was arbitrary. □

As we saw in the previous section, solutions of (6.1) for fixed u, do not necessarily have constant momentum (a similar technicality occurs for the Burgers equations, as discussed in the previous chapter). This added some complications to the proof of the well-posedness for these systems in $H^{1/2}(\mathbb{T}^3)$, namely that we needed to use the form of the equations to estimate inhomogeneous Sobolev norms with homogeneous ones. However, in the case of (6.19), like the Navier–Stokes equations, initial data with zero average gives rise to solutions that also have this property for positive times. To see this formally, we integrate (6.19) over $\mathbb{T}^3$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} w \, \mathrm{d}x = -\int_{\mathbb{T}^3} ((\mathbb{P}w) \cdot \nabla) w + \frac{1}{2} \nabla |w|^2 - \Delta w \, \mathrm{d}x = 0$$

where the first term on the right-hand side vanishes because $\mathbb{P}w$ is weakly divergence free and the other terms vanish by periodicity. For this reason, in what follows, we will prove well-posedness for solutions in certain homogeneous Sobolev spaces $\dot{H}^s(\mathbb{T}^3)$.

As in the previous section we will at first consider a weak formulation of (6.19). We call $w \in L^\infty(0, T; L^2) \cap L^2(0, T; \dot{H}^1)$ a weak solution of (6.19) with initial data $w_0 \in L^2$ if

$$\int_0^t (((\mathbb{P}w)\cdot\nabla)w(s) + (\nabla w)^\top w, \phi(s))_{L^2} + (\nabla w(s), \nabla\phi(s))_{L^2}\,\mathrm{d}s$$
$$= (w_0, \phi(0))_{L^2} - (w(t), \phi(t))_{L^2} + \int_0^t (w(s), \partial_t\phi(s))_{L^2}\,\mathrm{d}s \quad (6.22)$$

for all $\phi \in C_c^\infty([0,T) \times \mathbb{T}^3)$ and all $t \in [0,T)$. If w has the additional regularity $w \in L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$ and $\partial_t w \in L^2(0,T;H^{-1/2})$, we say that w is an $H^{1/2}$ solution.

It is not clear whether weak solutions of (6.19) can be found directly for L^2 data, as they can for the Navier–Stokes equations. Indeed, the second nonlinear term does not seem amenable to the necessary energy estimates if we only have $w_0 \in L^2$. However for $w_0 \in \dot{H}^{1/2}$ we will show that there exists a unique weak $H^{1/2}$ solution on $[0,T]$ for some $T > 0$. Moreover, we will show that the solution becomes smooth, immediately after the initial time, and can be extended to solutions on $[0,\infty)$.

Again, we will prove well-posedness of $H^{1/2}$ solutions using Galerkin approximations. For fixed $w_0 \in \dot{H}^{1/2}$ we denote by $w_n \in C^\infty([0,T_n) \times \mathbb{T}^3)$ the solution of the truncated equation

$$\frac{\partial}{\partial t}w_n + P_n\left[((\mathbb{P}w_n)\cdot\nabla)w_n + \frac{1}{2}\nabla|w_n|^2\right] - \Delta w_n = 0 \quad (6.23)$$

with initial data $P_n w_0$. Here $T_n > 0$ is the maximal existence time for the solution w_n, of this system of ODEs.

We can now state the main result of this section.

Theorem 6.9. *Given $w_0 \in \dot{H}^{1/2}(\mathbb{T}^3)$ there exists a unique global $H^{1/2}$ solution of (6.19), $w \in C([0,\infty);\dot{H}^{1/2}) \cap L^2(0,\infty;\dot{H}^{3/2})$, such that $w(0) = w_0$. Moreover $w \in C^1((0,\infty);C(\mathbb{T}^3)) \cap C((0,\infty);C^2(\mathbb{T}^3))$ is a classical solution, except at time $t = 0$.*

The proof is analogous to our analysis of the Burgers equations in the last chapter. As we did in that case we divide the proof of Theorem 6.9 into the following two lemmas.

Lemma 6.10. *If $w_0 \in \dot{H}^1(\mathbb{T}^3)$ there exists a unique global solution of (6.19) $w \in C([0,\infty);\dot{H}^1) \cap L^2(0,T;\dot{H}^2)$ such that $w(0) = w_0$. Moreover*

$w \in C^1((0,\infty);C(\mathbb{T}^3)) \cap C((0,\infty);C^2(\mathbb{T}^3))$ is a classical solution, except possibly at time $t = 0$.

Lemma 6.11. *For any $w_0 \in \dot{H}^{1/2}$ there exists a unique $H^{1/2}$ solution of (6.19) on $[0, T)$ for some $T > 0$.*

In the proof of Lemma 6.10 we will obtain local well-posedness and smoothness in the same way as we can for the Navier–Stokes equations. Global well-posedness then follows, using estimates that make use of a maximum principle (Lemma 6.8). To prove Lemma 6.11, we use analogous arguments to those in the previous chapter. That is, splitting the initial data from the nonlinearity, similar to the argument in Calderón (1990) (see also Chemin et al. (2006) or Marín-Rubio et al. (2013) for expositions).

6.3.1 Proof of Lemma 6.10

First, note that the formal proof that the full system conserves momentum can be adapted to an approximation w_n satisfying (6.23). Hence

$$\int_{\mathbb{T}^3} w_n(x,t)\,\mathrm{d}x = \int_{\mathbb{T}^3} P_n w_0(x)\,\mathrm{d}x = 0.$$

Integrating (6.23) against $2\Lambda^2 w_n$, and proceeding as for strong solutions of the Navier–Stokes equations (see Robinson et al. (2016), for example), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_n\|_1^2 + \|w_n(t)\|_2^2 \le c\|w_n(t)\|_1^6 \tag{6.24}$$

for all $t \in [0, T_n]$ and some $c > 0$. Considering only the terms in $\|w_n\|_1^2$ and solving the resulting differential inequality, we obtain

$$\|w_n(t)\|_1^2 \le \frac{\|P_n w_0\|_1^2}{\sqrt{1 - 2ct\|P_n w_0\|_1^4}}.$$

Fixing $T < (2c\|w_0\|_1^4)^{-1}$, it follows from maximality of T_n that $T_n > T$ and $\|w_n(t)\|_1$ is bounded, independent of n, on $[0, T)$. From (6.24), it then follows that w_n is uniformly bounded in $L^2(0, T; \dot{H}^2)$.

Using these uniform bounds we have the following estimates on the nonlinear terms from (6.23) in $L^2(0,T;L^2)$:

$$\left(\int_{\mathbb{T}^3} |(\nabla w_n)^\top w_n|^2\right)^{1/2} \le \left(\int_{\mathbb{T}^3} |\nabla w_n|^3\right)^{1/3} \left(\int_{\mathbb{T}^3} |w_n|^6\right)^{1/6}$$
$$\le c\|w_n\|_{H^{3/2}}\|w_n\|_{H^1} \in L^2(0,T).$$

A similar estimate holds for the other nonlinear term, hence $\partial_t w_n$ is uniformly bounded in $L^2(0,T;L^2)$. By the Aubin–Lions lemma, there exists a subsequence w_n converging to $w \in L^2(0,T;\dot{H}^1)$. Moreover, w is a weak solution of (6.22), $\partial_t w \in L^2(0,T;L^2)$ and $w \in C([0,T);\dot{H}^1) \cap L^2(0,T;\dot{H}^2)$.

To prove that w is a classical solution after the initial time, we have the following lemma. We omit the proof because it is very similar to arguments applicable to the Navier–Stokes equations which are described in Constantin and Foias (1988) and Robinson (2006b), for example. See also Lemma 4.5.

Lemma 6.12. *If the sequence $(w_n)_{n=1}^\infty$ is bounded in $L^2(\varepsilon,T;\dot{H}^s)$ for some $s > 3/2$ and some $\varepsilon \ge 0$ such that $\|w_n(\varepsilon)\|_s < \infty$, then they are also bounded uniformly in $L^\infty(\varepsilon,T;\dot{H}^s) \cap L^2(\varepsilon,T;\dot{H}^{s+1})$.*

Applying this lemma five times, we see that $(w_n)_{n=1}^\infty$ is a bounded sequence in $L^\infty(\varepsilon,T;\dot{H}^6)$ for all $\varepsilon \in (0,T)$. Using the Banach algebra property of H^s for $s > 3/2$, this gives us the following estimates on the time derivatives of w_n:

$$\sup_{t\in(\varepsilon,T)} \left\|\frac{\partial w_n}{\partial t}(t)\right\|_4 \le c \sup_{t\in(\varepsilon,T)} (\|w_n(t)\|_4\|w_n(t)\|_5 + \|w_n(t)\|_6)$$

and (differentiating (6.23))

$$\sup_{t\in(\varepsilon,T)} \left\|\frac{\partial^2 w_n}{\partial t^2}(t)\right\|_2 \le c \sup_{t\in(\varepsilon,T)} \left(\left\|\frac{\partial w_n}{\partial t}(t)\right\|_4 + \left\|\frac{\partial w_n}{\partial t}(t)\right\|_2 \|w_n(t)\|_3 + \left\|\frac{\partial w_n}{\partial t}(t)\right\|_3 \|w_n(t)\|_2\right).$$

Therefore w_n is uniformly bounded in $H^2(\varepsilon,T;\dot{H}^2) \cap H^1(\varepsilon,T;\dot{H}^4)$. This regularity passes to the limit (see Corollary 1.10); hence by the appropriate embeddings (see Corollary 1.13)

$$w \in C^1([\varepsilon,T];C^0) \cap C([\varepsilon,T];C^2)$$

is a classical solution on $[\varepsilon, T]$. Note that we may consider a closed interval by using the above argument on a larger open interval.

Since w is a classical solution we can apply Lemma 6.8 to obtain

$$\sup_{t\in[\varepsilon,T]} \|w(t)\|_{L^\infty} \leq \|w(\varepsilon)\|_{L^\infty}.$$

This allows the following additional H^1 estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_1^2 \leq |(2((\mathbb{P}w)\cdot\nabla)w + \nabla|w|^2, -\Delta w)_{L^2}| - 2\|w\|_2^2 \leq c\|w\|_{L^\infty}^2\|w\|_1^2. \tag{6.25}$$

Notice that care must be taken with the first nonlinear term because $\mathbb{P}w$ is an unbounded operator on L^∞. We therefore argue using the anti-symmetry, $(\mathbb{P}w\cdot\nabla v_1, v_2)_{L^2} = -(\mathbb{P}w\cdot\nabla v_2, v_1)_{L^2}$, as follows:

$$\begin{aligned}&((\mathbb{P}w\cdot\nabla)w, -\partial_{xx}w)_{L^2} = (\partial_x[(\mathbb{P}w\cdot\nabla)w], \partial_x w)_{L^2}\\ &= ((\mathbb{P}\partial_x w\cdot\nabla)w, \partial_x w)_{L^2} + ((\mathbb{P}w\cdot\nabla\partial_x w), \partial_x w)_{L^2} = -((\mathbb{P}\partial_x w\cdot\nabla)\partial_x w, w)_{L^2},\end{aligned}$$

for any spatial derivative ∂_x. Hence the inequality

$$|((\mathbb{P}w)\cdot\nabla)w, -\Delta w)_{L^2}| \leq \|w\|_1\|w\|_2\|w\|_{L^\infty},$$

holds, in the absence of L^∞ bounds on $\mathbb{P}w$.

From (6.25) and Lemma 6.8, it follows that for all $t \in [0, T)$

$$\|w(t)\|_1^2 \leq \|w_0\|_1^2 \mathrm{e}^{ct\|w_0\|_{L^\infty}^2}.$$

This rules out the finite-time blowup of $\|w(t)\|_1$, therefore since we can extend a solution on $[0, T)$ onto $[0, T+\delta)$ where $\delta \propto \|w(T)\|_1^{-4}$, there exists a solution $w \in C^1([0,\infty); C^0) \cap C([0,\infty); C^2)$.

We have now proved that for initial data in $\dot{H}^1$ there exists a global weak solution to (6.19) that is classical, except possibly at the initial time. To complete the proof of Lemma 6.10 it remains to show that these solutions are unique. The following lemma also shows that even less regular solutions are unique and will be useful in the next section.

Lemma 6.13. *If w_1, $w_2 \in L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$ are $H^{1/2}$ solutions of (6.22) corresponding to the same initial data $w_0 \in \dot{H}^{1/2}$ then $w_1 = w_2$.*

Proof. Fix $S \in [0,T)$ and let $(\psi_n)_{n=1}^\infty \subset C_c^\infty([0,T) \times \mathbb{T}^3)$ be a sequence of test functions such that $\int_{\mathbb{T}^3} \psi_n(t) = 0$ for each $t \in [0,T)$, and

$$\psi_n \to \Lambda^{1/2}(w_1 - w_2) \text{ in } L^2(0,S;\dot{H}^1) \text{ and } C([0,S];L^2),$$

with $\partial_t \psi_n \to \partial_t(w_1 - w_2)$ in $L^2(0,S;H^{-1})$. We may construct such a sequence using Lemma 1.15. Set $\phi_n := \Lambda^{1/2}\psi_n$ in (6.22), then the difference $w_1 - w_2$ satisfies

$$\begin{aligned}
(\Lambda^{1/2}(w_1 - w_2)(S), \psi_n(S))_{L^2} &+ \int_0^S (\Lambda^{1/2}\nabla(w_1 - w_2), \nabla\psi_n) \\
&- \int_0^S (\Lambda^{1/2}(w_1 - w_2), \partial_t\psi_n(s)) \\
&\le c\int_0^S (\|w_1 - w_2\|_{1/2}\|w_1\|_{3/2} + \|w_2\|_1\|w_1 - w_2\|_1)\|\psi_n\|_1 \\
&+ c\int_0^S (\|w_1 - w_2\|_1\|w_1\|_1 + \|w_2\|_{3/2}\|w_1 - w_2\|_{1/2})\|\psi_n\|_1 \\
&\le c\int_0^S (\|w_1\|_{3/2} + \|w_2\|_{3/2})\|w_1 - w_2\|_{1/2}\|\psi_n\|_1 \\
&+ c\int_0^S (\|w_1\|_1 + \|w_2\|_1)\|w_1 - w_2\|_{1/2}^{1/2}\|w_1 - w_2\|_{3/2}^{1/2}\|\psi_n\|_1 \\
&\le c\int_0^S (\|w_1\|_{3/2} + \|w_2\|_{3/2})^2\|w_1 - w_2\|_{1/2}^2 + \frac{1}{4}\int_0^S \|\psi_n\|_1^2 \\
&+ c\int_0^S (\|w_1\|_1 + \|w_2\|_1)^4\|w_1 - w_2\|_{1/2}^2 + \frac{1}{4}\int_0^S \|w_1 - w_2\|_{3/2}^{2/3}\|\psi_n\|_1^{4/3}
\end{aligned}$$

for every n. Letting $n \to \infty$, the left-hand side converges to

$$\frac{1}{2}\|(w_1 - w_2)(S)\|_{1/2}^2 + \int_0^S \|w_1 - w_2\|_{3/2}^2,$$

so we obtain

$$\begin{aligned}
&\|(w_1 - w_2)(S)\|_{1/2}^2 \\
&\qquad \le c\int_0^S (\|w_1\|_{3/2}^2 + \|w_2\|_{3/2}^2 + \|w_1\|_1^4 + \|w_2\|_1^4)\|w_1 - w_2\|_{1/2}^2.
\end{aligned}$$

Since the parenthesised part in the integral belongs to $L^1(0,T)$, Gronwall's Lemma now implies that $\|(w_1 - w_2)(S)\|_{1/2} = 0$. Since $S \in [0,T)$ was arbitrary, we deduce that $w_1 = w_2$, as required. $\square$

6.3.2 Proof of Lemma 6.11

In this section we prove the local well-posedness of (6.19) with initial data $w_0 \in \dot{H}^{1/2}$. Uniqueness follows from Lemma 6.13, so it suffices to prove local existence of $H^{1/2}$ solutions.

Following the arguments in the previous chapter (see also Calderón (1990), Marín-Rubio et al. (2013) and Pooley and Robinson (2016a)), we find the necessary estimates by decomposing the Galerkin approximations w_n, which solve (6.23), into a sum $w_n = v_n + z_n$ where

$$\begin{cases} \partial_t v_n - \Delta v_n = 0 \\ \quad v_n(0) = P_n w_0 \end{cases}$$

and

$$\begin{cases} \partial_t z_n - \Delta z_n = -P_n \left[((\mathbb{P} w_n) \cdot \nabla) w_n + \dfrac{1}{2} \nabla |w_n|^2 \right] \\ \hfill z_n(0) = 0. \end{cases} \tag{6.26}$$

From the heat equation satisfied by v_n, it is easy to check that for any $t \geq 0$ and any n

$$\|v_n(t)\|_{1/2}^2 + 2 \int_0^t \|v_n(s)\|_{3/2}^2 \, \mathrm{d}s \leq \|P_n w_0\|_{1/2}^2, \tag{6.27}$$

hence $v_n \in L^\infty(0, T; H^{1/2}) \cap L^2(0, T; H^{3/2})$ is uniformly bounded (independent of n and t). It therefore suffices to find estimates on z in the same spaces.

Integrating (6.26) against Λz_n yields

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|z_n(t)\|_{1/2}^2 + \|z_n(t)\|_{3/2}^2 &\leq |(((\mathbb{P} w_n) \cdot \nabla) w_n + (\nabla w_n)^\top w_n, \Lambda^1 z_n)_{L^2}| \\ &\leq c \|w_n(t)\|_1^2 \|z_n(t)\|_{3/2} \\ &\leq c \|v_n\|_1^4 + \frac{1}{2} \|z_n\|_{3/2}^2 + \|z_n\|_{3/2}^2 \|z_n\|_{1/2}, \end{aligned}$$

where we have used the same Sobolev embeddings and interpolations for $H^{1/2}$ and $H^{3/2}$ as we did in Section 6.2.

After simplifying this inequality and integrating over $[0,t]$, we have

$$\|z_n(t)\|_{1/2}^2 + \int_0^t \|z_n(s)\|_{3/2}^2 \,\mathrm{d}s \le c\int_0^t \|v_n(s)\|_1^4 \,\mathrm{d}s + \frac{1}{2}\sup_{s\in[0,t]} \|z_n(s)\|_{1/2}^2 + \frac{1}{2}\left(\int_0^t \|z_n(s)\|_{3/2}^2 \,\mathrm{d}s\right)^2,$$

where we have applied Young's inequality to the last term.

Since $\|v_n\|_1 \le \|v\|_1$ where v solves

$$v_t - \Delta v = 0,\ v(0) = w_0,$$

and $v \in L^4(0,t;\dot H^1)$ for all $t > 0$ we can make the first term on the right-hand side independent of n. This gives an estimate of the form

$$\sup_{s\in[0,t]} f_n(s) + \int_0^t g_n(s)\,\mathrm{d}s \le A(t)\sup_{s\in[0,t]} f_n(s) + B(t)\left(\int_0^t g_n(s)\,\mathrm{d}s\right)^2 + C(t),$$

where $f_n(t) \coloneqq \|z_n(t)\|_{1/2}^2$,

$$g_n(t) \coloneqq \int_0^t \|z_n(s)\|_{3/2}^2 \,\mathrm{d}s,$$

$A \equiv 0$, $B \equiv \frac{1}{2}$, and

$$C(t) \coloneqq c\int_0^t \|v_n(s)\|_1^4 \,\mathrm{d}s.$$

This allows us to apply Lemma 5.6 to find $T > 0$ independent of n such that f_n and g_n are uniformly bounded.

It follows that w_n uniformly bounded in $L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$ independent of n. As in the existence argument from Section 6.2 we also have bounds on $\frac{\partial}{\partial t}w_n \in L^2(0,T;H^{-1/2})$, independent of n. Therefore, by the Aubin–Lions lemma, passing to a subsequence we may assume that w_n converges in $L^2(0,T;H^{1/2})$ to a limit w that is an $H^{1/2}$ solution of (6.19).

Since for all $\varepsilon > 0$ there exists $t \in (0,\varepsilon)$ such that $w(t) \in \dot H^1$, Lemma 6.10 implies that w is a classical solution on $(0,T)$ and can be extended to a classical solution on $(0,\infty)$. By Lemma 6.13 this solution is unique. Thus Theorem 6.9 is proved.

6.4 Conclusions

We have reviewed how the Navier–Stokes equations can be reformulated using a magnetisation variable:

$$w_t + (u \cdot \nabla)w + (\nabla u)^\top w - \nu \Delta w = 0 \tag{6.28}$$

$$u = \mathbb{P}w.$$

The two systems were known to be equivalent in the sense of classical solutions. We discussed how the systems correspond in the setting of weak and $H^{1/2}$ solutions. In particular, we showed that a weak solution of (6.28) gives rise to unique weak solution of the Navier–Stokes equations. Conversely weak solutions of the Navier–Stokes equations correspond to families of functions that satisfy a weak version of (6.28) but only when tested against divergence-free functions.

We then proved that for a more regular weak solution of the Navier–Stokes equations $u \in L^\infty(0,T;H^{1/2}) \cap L^2(0,T;H^{3/2})$, there exists a unique $H^{1/2}$ solution of (6.28) (subject to the choice of initial data such that $\mathbb{P}w_0 = u_0$).

In Section 6.3 we proved global well-posedness and regularity results for a system obtained by replacing the second nonlinear term $(\nabla \mathbb{P}w)^\top w$ with $(\nabla w)^\top w = \frac{1}{2}\nabla|w|^2$. The new system (6.19) exhibited conservation of momentum, like the Navier–Stokes equations, but admitted a simple maximum-principle, like the Burgers equations.

In view of these results, it would be interesting to investigate the well-posedness, or otherwise, of a system

$$w_t + (\nabla \mathbb{P}w)^\top w - \Delta w = 0 \tag{6.29}$$

obtained by taking only the nonlinear term from (6.28) that was replaced in (6.19) because it caused the proof of the maximum principle to fail.

Chapter 7

Conclusions

7.1 Summary of results

In this thesis we have studied well-posedness questions for the Euler and Navier–Stokes equations via alternative formulations. For the Euler equations we gave a new local well-posedness proof in $H^s(\mathbb{T}^d)$ for $s > 1 + d/2$, using the Eulerian-Lagrangian formulation, discussed by Constantin. We also gave a careful derivation of that formulation, considering classical solutions of the Euler equations and of the reformulation, which we now recall:

$$\partial_t A + (u \cdot \nabla)A = 0,\ \ A(0, x) = x,$$

$$\partial_t v + (u \cdot \nabla)v = 0,\ \ v(0, x) = u_0(x),$$
$$u = \mathbb{P}((\nabla A)^\top v).$$

We derived a similar formulation of the Navier–Stokes equations that included the proper back-to-labels map, rather than a diffusive analogue:

$$\partial_t A + (u \cdot \nabla)A = 0,\ \ A(0, x) = x,$$

$$\partial_t v + (u \cdot \nabla)v - [(\nabla A)^{-1}]^\top \Delta[(\nabla A)^\top v] = 0,\ \ v(0, x) = u_0(x),$$
$$u = \mathbb{P}((\nabla A)^\top v).$$

Considering that formulation and, in particular, the variable $w \coloneqq (\nabla A)^\top v$ we obtain the magnetization variables formulation previously studied by

Montgomery-Smith and Pokorny:

$$\partial_t w + (u \cdot \nabla)w + (\nabla u)^\top w - \Delta w = 0,\ w(0) = w_0 \text{ such that } \mathbb{P}w_0 = u_0,$$

where

$$u = \mathbb{P}w.$$

We noted the similarities between the magnetization variables formulation and the diffusive Burgers equations, in particular the lack of an incompressibility constraint on w. In Chapters 4 and 5 we gave proofs of global well-posedness for the Burgers equations in several spaces, including L^p for $p > d$ in bounded domains and the whole space, and $H^{1/2}(\mathbb{T}^3)$. We also commented that it seems reasonable to expect, given the a priori estimates, that with a little more care we can prove global well-posedness in L^3. Our proofs were adapted from well-known analyses of the Navier-Stokes equations, but were complicated by the absence of an incompressibility constraint, and the consequent non-conservation of momentum and lack of an L^2 theory.

In Chapter 6 we discussed the direct derivation of the magnetization variables formulation of the Navier–Stokes equations and the relationship between the two systems for weak solutions $w \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$ and equivalence for solutions in $L^\infty(0,T;H^{1/2}(\mathbb{T}^3)) \cap L^2(0,T;H^{3/2}(\mathbb{T}^3))$.

We then observe that by slightly modifying one of the nonlinear terms, we obtain a model system that admits a maximum principle:

$$\partial_t w + (\mathbb{P}w \cdot \nabla)w + (\nabla w)^\top w - \Delta w = 0.$$

We proved that this system is globally well-posed in $H^{1/2}(\mathbb{T}^3)$, following our analysis of the Burgers equations. Indeed the model system exhibits momentum conservation, so our arguments are slightly more straightforward, although there is still no L^2 theory to make use of.

In the next section we will present some of the outstanding questions arising from the discussions in this thesis and suggest topics for future work.

7.2 Suggestions for future work

In Chapter 3, we proved local well-posedness for the Eulerian-Lagrangian formulation of the Euler equations in $H^s(\mathbb{T}^d)$ where $s > 1+d/2$, and $d \geq 2$. It would be worthwhile extending this result to solutions on $\mathbb{R}^d$ or even bounded domains.

In the case of unbounded domains we would need to find alternatives for the estimates that relied on compactness of the domain. In the case of a bounded domain there is the difficulty that Lagrangian trajectories may reach the boundary, so it may be more difficult to prove that the back-to-labels map is meaningful and find suitable estimates in that case.

Also in Chapter 3 we derived the following Eulerian-Lagrangian formulation of the Navier–Stokes equations:

$$\partial_t A + (u \cdot \nabla)A = 0,\ A(x,0) = x, \tag{7.1}$$

$$u = \mathbb{P}[(\nabla A)^\top v] \tag{7.2}$$

$$\partial_t v + (u \cdot \nabla)v - \left[(\nabla A)^\top\right]^{-1} \Delta \left[(\nabla A)^\top v\right] = 0,\ v(x,0) = u_0(x). \tag{7.3}$$

We suggested a family of models for this system, where (7.3) is replaced by

$$\partial_t v + (u \cdot \nabla)v - M^{-1}\Delta\,[Mv] = 0,\ v(x,0) = u_0(x), \tag{7.4}$$

for a fixed family of invertible matrices $M(x,t)$ (with or without the dependence on t). If M is orthogonal for all (x,t) then this model system seems amenable to the same estimates that we used to prove local well-posedness for the Eulerian-Lagrangian formulation of the Euler equations in $H^s(\mathbb{T}^3)$ ($s > 1 + d/2$). This discussion leaves open the following problems (among others):

1. Prove (local) existence and uniqueness for (7.1–7.3) in H^s in the case $s > 1 + d/2$ using an approach analogous to our treatment of the Eulerian-Lagrangian formulation of the Euler equations.

2. Find necessary and sufficient conditions on M such that (7.1, 7.2, 7.4) is locally well posed in $H^s(\mathbb{T}^d)$ ($s > 1 + d/2$).

3. Investigate the relationship between the regularity of u and v in either system. If we can expect that v is more regular than its counterpart in

the non-diffusive case, determine whether this can be used to weaken the hypothesis on u_0 required to prove local well-posedness.

Regarding our discussion of the Burgers equations, the following additional results may not be too difficult to obtain, by adapting the proofs of similar results for the Navier–Stokes equations, and making use of the maximum principle.

1. Prove global existence and uniqueness of weak solution with initial data in $L^p(\mathbb{R}^d)$ for $p > d$ (i.e. without the additional assumption $u_0 \in L^2(\mathbb{R}^d)$).

2. In three dimensions, prove global existence and uniqueness in $L^3(\Omega)$ for $\Omega = \mathbb{R}^3$, a bounded domain or $\mathbb{T}^3$.

3. In three dimensions, prove global existence and uniqueness of weak solutions with initial data in $H^{1/2}(\mathbb{R}^3)$ and likewise for data in $H_0^{1/2}(\Omega)$ (the closure in $H^{1/2}(\mathbb{R}^3)$ of C_c^∞ functions supported in Ω) for smooth bounded domains $\Omega \subset \mathbb{R}^3$.

4. Check whether or not the well-posedness results we have found on the whole space also apply to other unbounded domains.

Another, problem that might be more difficult would be to study solutions of the Burgers equations in L^2, since it is not clear how to estimate the energy of solutions without additional regularity. The lack of such estimates is a significant difference between the Burgers and Navier–Stokes equations.

We formalise this as the following problem: prove (or construct a counterexample to) global well-posedness for weak solutions of the Burgers equations for arbitrary initial data in L^2 (on any domain).

We considered the following model of the Navier–Stokes equations in magnetization variables:

$$\partial_t w + (\mathbb{P}w \cdot \nabla)w + \frac{1}{2}\nabla|w|^2 - \Delta w = 0, \tag{7.5}$$

and proved a global well-posedness result in $H^{1/2}(\mathbb{T}^3)$. It seems likely that more of our well-posedness results for the Burgers equations can be adapted to the model system. In particular it would be interesting to prove global well-posedness for (7.5) in $L^p(\Omega)$ for a domain $\Omega \subseteq \mathbb{R}^d$ or $\mathbb{T}^d$ with $p > d$ (or even $p = d$).

The magnetization-variables formulation may give rise to new conditions on a solution of the Navier–Stokes equations u to ensure regularity, or at least new proofs of known ones. More precisely, when considering the system

$$w_t + (u \cdot \nabla)w + (\nabla u)^\top w - \Delta w = 0, \tag{7.6}$$

the following problem is natural: find (necessary and) sufficient conditions on a fixed u such that a weak solution w of (7.6) is unique (or smooth).

Such a condition, would also be a sufficient to imply that a solution of the Navier–Stokes equations is unique (or smooth). For example, we might try to adapt the discussion of regularity of solutions of certain drift-diffusion equations in Silvestre and Vicol (2012).

In the context of the full magnetization-variables formulation (i.e. with $u = \mathbb{P}w$), it might also be worthwhile to follow the global well-posedness results by Friedlander and Vicol (2011) for drift diffusion systems with a linear coupling, of the form

$$w_t + (u \cdot \nabla)w - \Delta w = 0,\ \nabla \cdot u = 0,\ u = Lw$$

for L in a certain family of linear operators. That approach is based on DeGiorgi techniques. It would surely be very difficult to obtain a similar well-posedness results for (a reformulation of) the Navier–Stokes equations, but it would be interesting to see how far such methods can be adapted.

Considerations of this type might give an alternative approach to certain regularity criteria for the Navier–Stokes equations that are already known. For instance, the $L^\infty(0, T; L^3)$ endpoint case of the Serrin condition was solved relatively recently by Escauriaza, Seregin, and Šverák (2003), and the proof is somewhat difficult. It is conceivable that treating the Navier–Stokes equations as a linear system for the magnetization variables, without an incompressibility constraint on w (i.e. without a pressure term to estimate) would give rise to an alternative and perhaps simpler proof of that result.

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